We are starting "Taylor Series" and the department has its own notes on this topic called the "Taylor Notes"

Entry Task (skills needed for TN 1-3):

From Fall 2011 Final

Let $f(x) = 4x^2 - 5x + \ln(x)$.

- Find $f'(x)$, $f''(x)$, and $f'''(x)$.
- On the interval $\frac{3}{4} \le x \le \frac{5}{4}$, give the global maximum of f''(x) and f'''(x)

$f'(x) = 8x - 5 + \frac{1}{x}$

$f''(x) = 8 - \frac{1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$y = 8 - \frac{1}{x^2} \qquad \frac{3}{4} \le x \le \frac{5}{4}$

$8 - \frac{1}{(\frac{3}{4})^2} = 6.22$

$8 - \frac{1}{(\frac{5}{4})^2} = 7.36$

$\left|8 - \frac{1}{x^2}\right| \le 7.36$ ← Always true on interval

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$y = \frac{2}{x^3} \qquad \frac{3}{4} \le x \le \frac{5}{4}$

$\frac{2}{(\frac{3}{4})^3} = 4.74$

$\left|\frac{2}{x^3}\right| \le 4.74$ ← Bound

Visual: https://www.desmos.com/calculator/kawbzhu5rj

Visual: https://www.desmos.com/calculator/njwskv7izh

on the next pages we will use this…

Taylor Notes 1 (TN 1):

Quick Example and Definitions

Def'n: For a given function $y = f(x)$ at a given point $x = b$, we define

1st Taylor Polynomial

$$T_1(x) = f(b) + f'(b)(x - b)$$

1st Taylor Poly. Error Bound

If $|f''(x)| \leq M$ for all x, then

$$|f(x) - T_1(x)| \leq \frac{M}{2}|x - b|^2.$$

Example:

Let $f(x) = 4x^2 - 5x + \ln(x)$ at $x=1$

Give the 1st Taylor Polynomial and give a bound on the error if you use it on the interval $\frac{3}{4} \leq x \leq \frac{5}{4}$.

curvieness + distance from start effect error

$f(x) = 4x^2 - 5x + \ln(x) \approx T_1(x)$

$\approx \underbrace{f(1)}_{-1} + \underbrace{f'(1)}_{4}(x-1)$

$\approx$ $\boxed{1 + 4(x-1)}$

→ approximatley the same as $4x^2 - 5x + \ln(x)$

analysis tool!

$|f''(x)| = |8 - \frac{1}{x^2}| \leq 7.36$

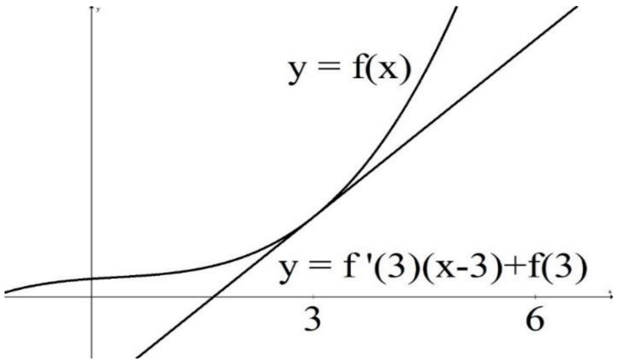
$|f(x) - T_1(x)| \leq \frac{7.36}{2}|x-1|^2$

$\frac{3}{4} \leq x \leq \frac{5}{4}$ $\quad \frac{5}{4} - 1 \quad \frac{3}{4} - 1$

$\text{error} \leq \frac{7.36}{2}\left(\frac{5}{4} - 1\right)^2 = \boxed{0.23}$

Visual: www.desmos.com/calculator/ymzw8fyys5

More Details



Tangent Linear Error Bound Thm

If $|f''(x)| \le M$ for all *x* values between *a* and *b*, then

ERROR = $|f(x) - T_1(x)| \le \frac{M}{2}|x-b|^2$.

Note:

M = some upper bound on f''(x)

$|x - b|$ = distance x is away from b.

To use the Tangent Line Error Bound:

1. Find f''(t).
2. Find upper bound for |f''(t)|.
3. Use the theorem.

Example:

Let f(x) = $x^{1/3}$ and b = 8.

(a) Find the 1st Taylor polynomial.

(b) Use it to approximate $\sqrt[3]{9}$.

(c) Give a bound on the error over the interval J = [7,9].

$x^{1/3} \approx 2 + \frac{1}{12}(x-8)$

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

$(9)^{1/3} \approx 2 + \frac{1}{12}(9-8) = 2.083$

$x^{\frac{1}{3}} - \left[2 + \frac{1}{12}(x-8)\right] \le$

$|f''(x)| = \frac{2}{9x^{5/3}} \le \frac{2}{9} \cdot \frac{1}{(7)^{5/3}}$

0.00867

error $\le \frac{}{2}|x-8|$

9 or 7

error $\le$ 0.0043

Proof sketch for x > b:

Start with $f(x) - f(b) = \int_b^x f'(t)dt$.

Integration by-parts in a clever way

(with $u = f'(t)$, $dv = dt$,

$du = f''(t)$, $v = t - x$)

$$f(x) - f(b) = f'(b)(x-b) - \int_b^x (t-x)f''(t)dt$$

$$f(x) - f(b) - f'(b)(x-b) = -\int (t-x)f''(t)dt$$

Thus, ERROR:

$|f(x) - T_1(x)| = \left|\int_b^x (x-t)f''(t)dt\right|$

$$\left|\int_b^x (x-t)f''(t)dt\right| \leq \int_b^x (x-t)|f''(t)|dt \leq M\int_b^x (x-t)dt \leq \frac{M}{2}(x-b)^2$$

Example: $f(x) = \ln(x)$ at $b = 1$.

(a) Find the 1st Taylor polynomial.

(b) Use the error bound formula to find a bound on the error over the interval *J* = [1/2, 3/2]

(c) Find an interval around b = 1 where the error is less than 0.01.

x	$f(x)$	$T_1(x)$	$\lvert f(x) - T_1(x)\rvert$
1	*0*	*0*	*0*
1.2	*0.1823*	*0.2*	*0.01768*
1.5	*0.4055*	*0.5*	*0.09453*
0.9	*-0.1053*	*-0.1*	*0.00536*
0.5	*-0.6931*	*-0.5*	*0.19314*

Note about "Bounds":

An **upper bound**, *M*, is a number that is *always* bigger than the function. The smallest possible upper bound is sometimes called a *tight* bound.

Examples: Find any upper **bound** (if it is easy to do so, find a *tight* upper bound).

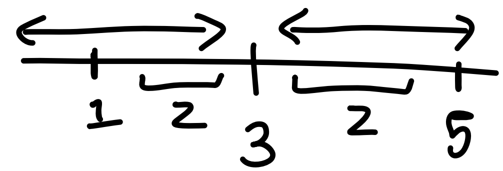
1. $|\sin(5x)|$ on $[0,2\pi]$

biggest sin can be is 1

$|\sin(5x)| \leq 1$

2. $|x-3|$ on $[1,5]$

$|x-3| \leq 2$



3. $\left|\frac{1}{(2-x)^3}\right|$ on $[-1,1]$

$\left|\frac{1}{(2-x)^3}\right| \leq 1$

4. $|\sin(x) + \cos(x)|$ on $[0,2\pi]$

$|\sin(x) + \cos(x)| \le 2$

$1 + 1 = 2$

5. $\left|\cos(2x) + e^{2x} + \frac{6}{x}\right|$ on $[1,4]$

$\le 1 + e^8 + 6$

HW

$y = e^x \quad -1 \le x \le 1$

$|f(x) - T_1(x)| \le \frac{e}{2}$

- $\frac{1}{2}$
- $\frac{e}{2}$
- $\frac{3}{2}$

TN 2: Quadratic Approximations

2nd Taylor Polynomial

$$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2$$

2nd Taylor Poly. Error Bound

If $|f'''(x)| \leq M$ for all *x*, then
$|f(x) - T_2(x)| \leq \frac{M}{6}|x-b|^3$.

Example (you do):
Let $f(x) = 4x^2 - 5x + \ln(x)$
Give the 2nd Taylor Polynomial and give a bound on the error if you use it on the interval $\frac{3}{4} \leq x \leq \frac{5}{4}$.

Visual: www.desmos.com/calculator/ymzw8fyys5

Example (from that same old exam)
Let $f(x) = 4x^2 - 5x + \ln(x)$
Give a value of *a* so that the error bound
on the interval
$1 - a \leq x \leq 1 + a.$
is less than or equal to 0.01.

Taylor Approximation Idea:
If two functions have ***all*** the same derivative values, then they are the same function (up to a constant).

To explain, let's compare derivatives of f(x) and T_2(x) at b.

$$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2} f''(b)(x-b)^2$$

$$T_2'(x) = 0 + \ f'(b) + \frac{1}{2} f''(b) 2(x-b)$$

$$T_2''(x) = 0 + 0 + f''(b)$$

$$T_2'''(x) = 0$$

Now plug in x = b to each of these.
What do you see?
Why did we need a ½ ?
What would T_3(x) look like?