Math 125D 3/8/24

Final Final Review

Final tomorrow: Kane 110, 1:30–4:20 PM

Bring TI-30X IIS, note sheet

Seat assignment on Canvas

Course evals due today

Your Name

Your Signature

Student ID #

Quiz Section

Professor's Name

TA's Name

- Turn off and stow away all cell phones, watches, pagers, music players, and other similar devices.
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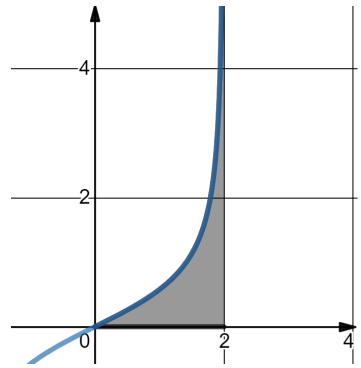
- You may use directly the integral formulas # 1-18 in the table from section 7.5 of your textbook (posted on the departmental math 125 website), without deriving them. **Show your work in evaluating any other integrals, even if they are on your note sheet**.
- Place $\boxed{\text{a box around your answer}}$ to each question. Unless otherwise instructed, simplify your answers, but leave them in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$).
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 9 pages, in addition to this cover sheet. Make sure you have a complete exam.

Question	Points	Score
1	12	
2	14	
3	8	
4	7	
5	21	

Question	Points	Score
6	8	
7	8	
8	10	
9	12	
Total	100	

5. Let $\mathcal{R}$ be the region in the first quadrant which is shown below, and it is described by:

$$0 \le y \le \frac{x}{\sqrt{4-x^2}}, \qquad 0 \le x < 2$$



Note that $f(x) = \frac{x}{\sqrt{4-x^2}}$ has a vertical asymptote; use limits for improper integrals as needed, and determine if they converge or diverge.

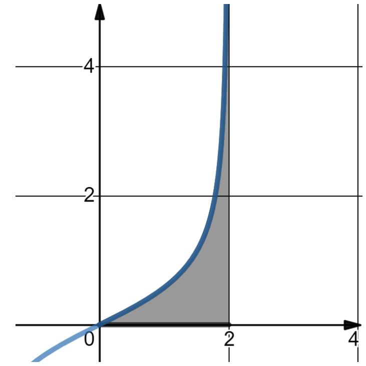
(a) (6 points) Compute the **area** of this region $\mathcal{R}$.

$$\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx = \lim_{t\to 2^-} \int_0^t \frac{x}{\sqrt{4-x^2}}\,dx$$

$u = 4-x^2 \quad du = -2x\,dx$

$$\lim_{t\to 2^-} \int_4^{4-t^2} \frac{-1}{2}\frac{du}{\sqrt{u}} = \lim_{t\to 2^-} \left(-\sqrt{u}\right)\Big]_4^{4-t^2} = \lim_{t\to 2^-}\left(\underbrace{-\sqrt{4-t^2}}_{0} + \sqrt{4}\right) = \boxed{2}$$

(b) (7 points) Compute the x-coordinate, $\bar{x}$, of its centroid (center of mass).



(c) (8 points) *Recall the region* $\mathcal{R}$ *from the previous page, bounded above by* $y = \dfrac{x}{\sqrt{4-x^2}}$, for $0 \le x < 2$.

Use limits for improper integrals as needed, and determine if they converge or diverge.

Compute the y-coordinate, $\bar{y}$, of the centroid of $\mathcal{R}$.

$$M_x = \int_0^2 \frac{1}{2}\frac{x^2}{4-x^2}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{2}\frac{x^2-4+4}{4-x^2}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{2}\left(-1+\frac{4}{4-x^2}\right)dx$$

Part. frac.:

$$\frac{4}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x}$$

$$4 = A(2+x) + B(2-x)$$

$x=2$: $4 = 4A$ $\quad A=1$

$x=-2$: $4=4B$ $\quad B=1$

$$= \lim_{t\to 2^-}\int_0^t \frac{1}{2}\left(-1+\frac{1}{2-x}+\frac{1}{2+x}\right)dx$$

$$= \lim_{t\to 2^-} \frac{1}{2}\left(-x - \ln|2-x| + \ln|2+x|\right)\Big]_0^t$$

$$= \lim_{t\to 2^-} \frac{1}{2}\left(-x + \ln\left|\frac{2+x}{2-x}\right|\right)\Big]_0^t$$

$$= \lim_{t\to 2^-} \frac{1}{2}\left(\underbrace{-t}_{-2} + \ln\underbrace{\left|\frac{2+t}{2-t}\right|}_{\to\infty} - \overbrace{\ln\left|\frac{2}{2}\right|}^{0}\right)$$

$\boxed{\text{diverges}}$

9. A 2000 L tank is full of a mixture of water and salt, with 500 grams of salt initial dissolved in the tank. Fresh water (with NO salt) is pumped into the tank at a rate of 20 L/s. The mixture is kept stirred and is pumped out at a rate of 40 L/s. (This means the tank is losing volume at a rate of 20 - 40 = -20 L/s).

   (a) (1 point) Give the linear function $V(t) = at + b$ for the volume in liters after t seconds.

$$V(t) = 2000 - 20t$$

   (b) (4 points) Let $y(t)$ be the amount of salt in grams in the tank after t seconds. Write down the differential equation AND initial condition satisfied by $y(t)$. Do not solve anything yet.

$$y' = \underbrace{0}_{\text{rate in}} - \underbrace{\overbrace{\frac{y}{2000-20t}}^{\text{prop. of salt}}\overbrace{(40)}^{\text{vol out}}}_{\text{rate out}}$$

Init cond: $y(0) = 500$

   (c) (6 points) Solve the differential equation to find $y(t)$. Show work. Simplify and box your answer.

$$y' = \frac{-2y}{100-t} \qquad \frac{dy}{dt} = \frac{-2y}{100-t} \qquad \int \frac{-1}{2y}\,dy = \int \frac{1}{100-t}\,dt$$

$$\frac{-1}{2}\ln|y| = -\ln|100-t| + C$$

$$\ln|y| = 2\ln|100-t| + D$$

note: y and $100-t$ are pos.

$$\ln(y) = 2\ln(100-t) + D$$

$$y = \underbrace{e^{2\ln(100-t)}}_{(e^{\ln(100-t)})^2}\underbrace{e^D}_{A}$$

$$y = A(100-t)^2$$

$t=0$, $y=500$

$$500 = A\cdot 100^2 \qquad A = \frac{1}{20}$$

$$\boxed{y = \frac{(100-t)^2}{20}}$$

   (d) (1 point) How many grams of salt are left in the tank after 60 seconds? Simplify your answer.

$$y = \frac{(100-60)^2}{20} = \frac{40^2}{20} = \boxed{80\text{ g}}$$

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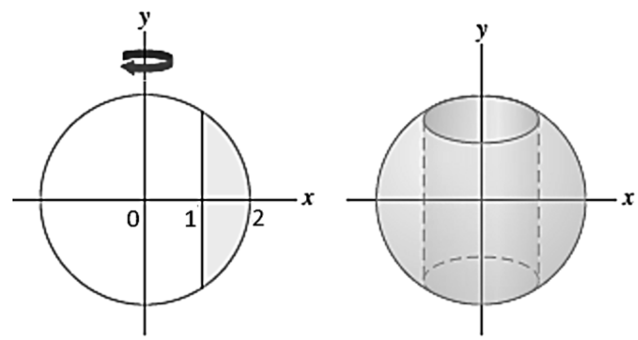
Table of Integration Formulas Constants of integration have been omitted.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$	**2.** $\int \frac{1}{x}\,dx = \ln\lvert x\rvert$
3. $\int e^x\,dx = e^x$	**4.** $\int b^x\,dx = \frac{b^x}{\ln b}$
5. $\int \sin x\,dx = -\cos x$	**6.** $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$	**8.** $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$	**10.** $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln\lvert \sec x + \tan x\rvert$	**12.** $\int \csc x\,dx = \ln\lvert \csc x - \cot x\rvert$
13. $\int \tan x\,dx = \ln\lvert \sec x\rvert$	**14.** $\int \cot x\,dx = \ln\lvert \sin x\rvert$
17. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	**18.** $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

- Place $\boxed{\text{a box around your answer}}$ to each question. Unless otherwise instructed, simplify your answers, but leave them in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$).
- All exam pages are double-sided except for this cover page and the last page. You may use the blank sides for extra room if needed but if you want us to grade these spare pages clearly **indicate in the problem area** that your work is on the back of the cover page or on the blank pages at the end of the exam.
- This exam has 10 problems on 9 pages. When the exam starts, make sure that your exam is complete. Good luck!

3. (5 points) A "bead" is formed by drilling a hole of radius $r = 1$ cm through the center of a sphere of radius $R = 2$ cm.
Set up an integral equal to the volume of the resulting bead. **Do not compute the integral.**



4. (5 points) Suppose all we know about some continuous function $g(x)$ is that

$$\frac{1}{x^2} \leq g(x) \text{ for all } x \geq 1.$$

Circle which of the following statements MUST be true, based on the provided information, and justify the statement(s) that you circled.

(a) $\int_1^\infty g(x)\,dx$ diverges $\int_1^\infty \frac{1}{x^2}\,dx$ conv.

(b) $\int_1^\infty xg(x)\,dx$ diverges $\int_1^\infty \frac{1}{x}\,dx$ div.

(c) $\int_1^\infty \frac{g(x)}{x}\,dx$ converges

(d) $\int_1^\infty g(x)\,dx$ converges

comp. test doesn't tell ~~you~~ whether the bigger func. must converge.

(e) None of the above

$\int_1^\infty \frac{1}{x^p}\,dx$ div. if and only if $p \leq 1$

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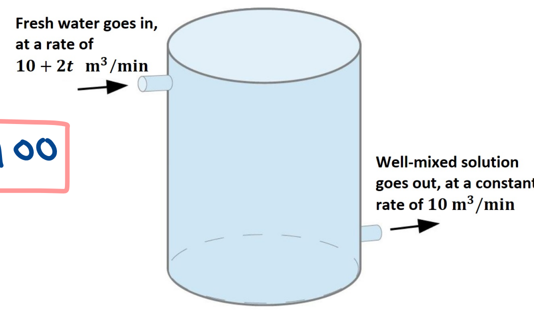
8. (14 points) At time $t = 0$ minutes a tank holds an initial volume $V_0 = 100$ m^3 of salty water, with an initial amount $S_0 = 3$ kg of salt dissolved in it.

Fresh water enters the tank at a rate of $10 + 2t$ m^3 per minute, where t is the time in minutes.

The salt always remains uniformly mixed throughout the water solution in the tank, and the solution exits the tank at a constant rate of 10 m^3/min.

(a) Find a formula for the volume $V(t)$ of salty water solution in the tank at time t minutes.

$$V'(t) = \underbrace{10+2t}_{\text{in}} - \underbrace{10}_{\text{out}} = 2t$$



$$V(t) = t^2 + \underbrace{C}_{100}$$ 100 b/c $V(0)=100$

$$\boxed{V(t) = t^2 + 100}$$

(b) Set up a differential equation for $S(t)$, which is the amount (in kg) of salt in the tank at t min. Do not solve it yet.

$$S' = 0 - \left(\frac{S}{t^2+100}\right)10 \to$$

$$\boxed{S(0)=3 \quad S' = \frac{-10S}{t^2+100}}$$

(c) Solve the differential equation in part (b) to find a formula for $S(t)$.

$$\frac{dS}{dt} = \frac{-10S}{t^2+100} \to \int \frac{-1}{10S}\,dS = \int \frac{dt}{t^2+100}$$

$$\frac{-1}{10}\ln|S| = \frac{1}{10}\arctan\left(\frac{t}{10}\right) + C$$

$$\ln|S| = -\arctan\left(\frac{t}{10}\right) + D$$

$$|S| = e^{-\arctan\left(\frac{t}{10}\right)} e^D$$

$$S = \underbrace{\pm e^D}_{A} e^{-\arctan\left(\frac{t}{10}\right)}$$

$$S = A e^{-\arctan\left(\frac{t}{10}\right)}$$

$3 = A \quad A = 3$

$$\boxed{S(t) = 3e^{-\arctan\left(\frac{t}{10}\right)}}$$

(d) How much salt is left in the tank after 10 min? Leave your answer in exact form.

$$S(10) = 3e^{-\arctan(1)} = \boxed{3e^{-\frac{\pi}{4}} \text{ kg}}$$