# Math 125D 3/4/24

Final Review

**DFEP #16 Solution:**

Let $I(t)$ be the amount of Iocaine powder in the jug of wine after $t$ minutes. We know that $I'(t) =$ (rate in) – (rate out). The rate in is simply 2 grams per minute. The rate out? Well, we're pouring out 0.5 liters (1/8 of the jug) per minute, so that's $I/8$. So we want to solve the differential equation:

$$\frac{dI}{dt} = 2 - \frac{I}{8} \qquad I(0) = 70$$

We can separate the differential equation to get

$$\frac{8\,dI}{16 - I} = dt$$

and integrate to get

$$\int \frac{8\,dI}{16 - I} = \int dt$$

$$-8 \ln |16 - I| = t + C$$

$$I = 16 + Ae^{\frac{-t}{8}}$$

Plugging in $t = 0$, $I = 70$ tells us that $A = 54$. So now we want to know when the wine is safe for Westley to drink, and when it's safe for everyone else. It's safe for Westley when there are 2 grams per 0.2 L, which means there are 40 grams in the 4 liter jug. Likewise, it's safe for everyone else when there are less than 0.1 grams per 0.2 L, which means there are 20 grams in the 4 liter jug.

Plugging in $I = 40$ and $I = 20$ and solving for $t$ yields $t = -8 \ln\left(\frac{24}{54}\right) = 6.49$ minutes when it becomes safe for him to drink, and $t = -8 \ln\left(\frac{4}{54}\right) = 20.82$ minutes when it becomes safe for everyone else.

So he should let this mixing process happen for any time between 6.49 and 20.82 minutes.

# Final Reminders:

1) Final is Saturday, 1:30-4:20 PM in KNE 110

Cumulative, list of topics on Canvas

2) Form for req. makeups in email

3) Seating chart pref. form on Canvas.

4) Course evals!

# A List of Topics for the Final

Here's what you should be able to do for the final exam.

## Really Old Stuff

1. Riemann sums
   (a) Compute $L_n$, $R_n$, and $M_n$ estimates for areas under curves.
   (b) Write the (exact) area under a curve as a limit of Riemann sums and (for certain curves) evaluate that limit.
   (c) Recognize such a limit, convert it to an integral, and compute it.
2. Integration
   (a) Find antiderivatives of certain elementary functions including polynomials, exponential functions, and certain trigonometric functions.
   (b) Use $u$-substitution to evaluate more challenging integrals.
   (c) Compute indefinite integrals and definite integrals.
   (d) Evaluate integrals of odd or even functions on intervals of the form $[-a, a]$.
   (e) Use the fundamental theorem of calculus to differentiate functions that are defined in terms of integrals.
3. Applications
   (a) Given velocity or acceleration, compute the net displacement of an object over a time interval *or* compute its total distance traveled.
   (b) Find the area bounded by two or more curves in the plane.

## Old Stuff

4. More applications
   (a) Compute the volumes of solids by integrating their cross-sectional areas.
   (b) In particular, use the washer method for finding volumes of solids of revolution by integrating along the axis of rotation.
   (c) Find volumes of solids of revolution using the shell method.
   (d) Compute the work required to perform certain tasks.
   (e) Find the average value of a function over an interval.
5. More integration techniques
   (a) Understand how to use trigonometric identities to compute integrals of the forms $\int \sin^m(x)\cos^n(x)\,dx$ or $\int \tan^m(x)\sec^n(x)\,dx$.
   (b) Know how and when to use the following techniques:
      - Integration by parts
      - Trigonometric substitution
      - Integration with partial fractions

6. Integral approximation
    1. (a) Approximate integrals with the trapezoid rule or Simpson's rule.
    2. (b) Know when $L_n$, $R_n$, $M_n$, or $T_n$ are underestimates or overestimates.
7. Improper integrals
    1. (a) Evaluate type-1 and type-2 improper integrals.
    2. (b) Use integral comparison to tell whether certain integrals converge or diverge, even when their integrands are hard to antidifferentiate explicitly.

## New Stuff

8. Arc length
    1. (a) Set up an integral to compute the arc length of a curve on some interval.
    2. (b) Approximate that integral if it's too hard to compute explicitly.
9. Center of mass
    1. (a) Compute moments around the $x$- and $y$-axes for shapes of uniform density.
    2. (b) Find the center of mass of such a shape.
    3. (c) Compute the centroid of a region.
10. Differential equations
    1. (a) Solve separable differential equations given initial conditions.
    2. (b) Set up differential equations for word problems, such as those involving mixing.

Your Name

Your Signature

Student ID #

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Quiz Section

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Professor's Name

TA's Name

- Turn off and stow away all cell phones, watches, pagers, music players, and other similar devices.
- This exam is closed book. You may use one 8.5″ × 11″ sheet of handwritten notes (both sides OK). Do not share notes.
- You can use only a Texas Instruments TI-30X IIS calculator. No other models are allowed.
- In order to receive credit, you must **show your work**. If you do not indicate the way in which you solved a problem, or if the work shown is incorrect or incomplete, you may get little or no credit for it, even if your answer is correct.
- You may use directly the integral formulas in the table below. You must **show your work in evaluating any other integrals**, even if they are on your sheet of notes.

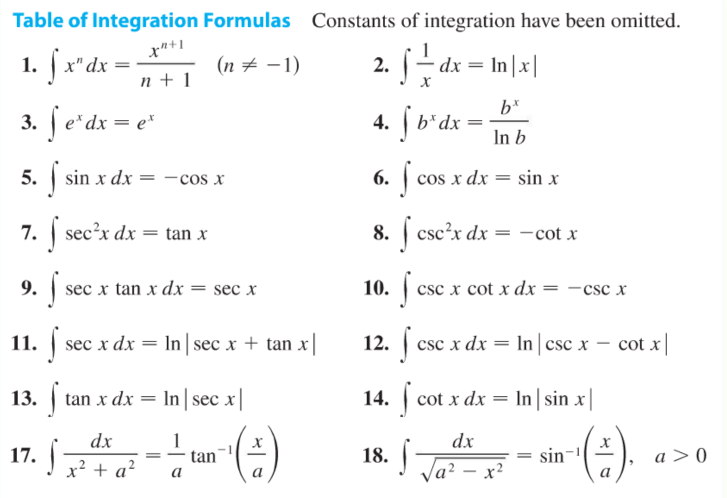


**Table of Integration Formulas** Constants of integration have been omitted.

| | |
|---|---|
| **1.** $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$ | **2.** $\int \frac{1}{x}\,dx = \ln\lvert x\rvert$ |
| **3.** $\int e^x\,dx = e^x$ | **4.** $\int b^x\,dx = \frac{b^x}{\ln b}$ |
| **5.** $\int \sin x\,dx = -\cos x$ | **6.** $\int \cos x\,dx = \sin x$ |
| **7.** $\int \sec^2 x\,dx = \tan x$ | **8.** $\int \csc^2 x\,dx = -\cot x$ |
| **9.** $\int \sec x \tan x\,dx = \sec x$ | **10.** $\int \csc x \cot x\,dx = -\csc x$ |
| **11.** $\int \sec x\,dx = \ln\lvert \sec x + \tan x\rvert$ | **12.** $\int \csc x\,dx = \ln\lvert \csc x - \cot x\rvert$ |
| **13.** $\int \tan x\,dx = \ln\lvert \sec x\rvert$ | **14.** $\int \cot x\,dx = \ln\lvert \sin x\rvert$ |
| **17.** $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ | **18.** $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$ |

- Place $\boxed{\text{a box around your answer}}$ to each question. Unless otherwise instructed, simplify your answers, but leave them in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$).
- All pages are double-sided except for this cover page and the last page. You may use the blank pages for extra room but, if you want us to grade these spare pages, clearly **indicate in the problem area** that your work is on the back of the cover page or on the blank page(s) at the end of the exam.
- This exam has 10 problems on 10 pages. When the exam starts, check that your exam is complete. Good luck!

1. (10 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int x\tan^2 x\,dx$ $= \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx$

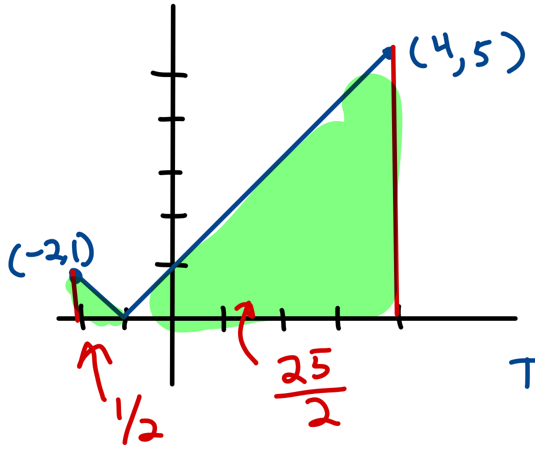
$u = x$ $\quad v = \tan x$

$du = dx$ $\quad dv = \sec^2 x\,dx$

$x\tan x - \int \tan x\,dx \quad - \frac{1}{2}x^2 = x\tan x - \frac{1}{2}x^2 - \int \tan x\,dx$

$$\boxed{= x\tan x - \frac{1}{2}x^2 - \ln|\sec x| + C}$$

(b) $\int_{-2}^{4} |x+1|\,dx$



Total = $\boxed{13}$

or:

$$\int_{-2}^{-1}(-x-1)\,dx + \int_{-1}^{4}(x+1)\,dx = \left(\frac{-1}{2}x^2 - x\right)\Big]_{-2}^{-1} + \left(\frac{1}{2}x^2 + x\right)\Big]_{-1}^{4}$$

$$\boxed{=13}$$

2. (10 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int \frac{1}{1+x^{2/3}}dx$ $= \int \frac{1}{1+u^2} 3u^2 du = \int \frac{3u^2}{1+u^2} du$

$u = x^{1/3}$

$du = \frac{1}{3} x^{-2/3} dx$

$3 \underbrace{x^{2/3}}_{u^2} du = dx$

or use long div.

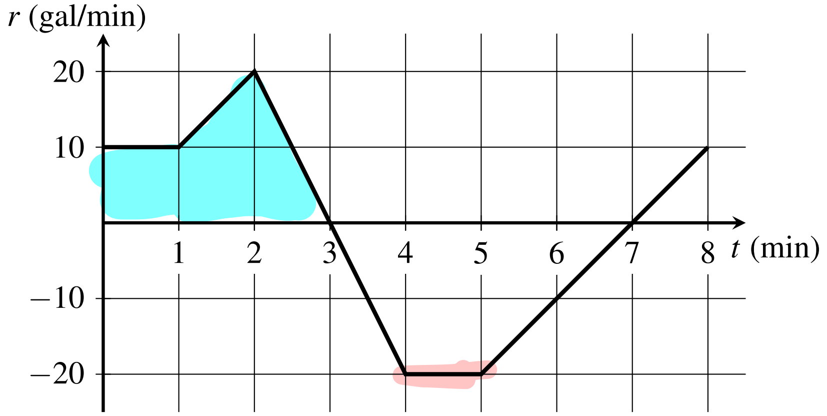
$= \int \frac{3u^2+3-3}{1+u^2} du = \int \left(3 - \frac{3}{1+u^2}\right) du$

$= 3u - 3\arctan(u) + C$

$\boxed{= 3x^{1/3} - 3\arctan(x^{1/3}) + C}$

(b) $\int x\sqrt{5+4x-x^2}\, dx$

3. (10 points) The following graph depicts a function $r(t)$ that gives the rate of flow of water into a tank, measured in gallons per minute. At time $t = 0$ minutes, the tank contains 100 gallons of water.



(a) What is the maximum amount of water in the tank at any time? Show work/justify.

Amount in tank at time $t$: $100+\int_0^t r(x)\,dx$

Max at $t=3$ (before decreasing) $\boxed{135 \text{ gal}}$

(b) At which time(s) is the amount of water in the tank decreasing the fastest?

$\boxed{t=4 \text{ to } t=5}$

(c) At which time(s) does the tank have the least amount of water in it?

$\boxed{t=7}$

(d) What is the average rate of flow of water into the tank over the first 4 minutes? Show work.

$\frac{1}{4}\int_0^4 r(t)\,dt$ $\boxed{\frac{25}{4} \text{ gal/min}}$

(e) Let $f(x) = \int_1^{x^2} r(t)\,dt$. Find $f'(2)$, showing your steps.

$f'(x) = r(x^2)(2x)$ $f'(2) = r(4)\cdot 4 = (-20)4 = \boxed{-80}$

4. (10 points) A high-speed bullet train travels between two consecutive stations that are 30 km apart. Find the time it takes the train to travel between these two stations, if the train starts at rest at the first station, accelerates at 10 km/min$^2$ until it reaches its maximum cruising speed of 3 km/min, drives at that speed for as long as possible, then decelerates at 5 km/min$^2$ in time to stop at the second station.

Accelerating: accelerates for $\boxed{\frac{3}{10}\text{ min}}$:

velocity is $\int 10\,dt = 10t + C, \quad C = 0$

dist traveled: $\int_0^{3/10} 10t\,dt = 5t^2\Big]_0^{3/10} = 5\cdot\frac{9}{100} = \boxed{\frac{9}{20}\text{ km}}$

Deceleration (but backwards): takes $\boxed{\frac{3}{5}\text{ min}}$

dist traveled = $\int_0^{3/5} 5t\,dt = \frac{5}{2}t^2\Big]_0^{3/5} = \boxed{\frac{9}{10}\text{ km}}$

Cruising dist: $30 - \left(\frac{9}{20} + \frac{9}{10}\right) = \frac{600-9-18}{20} = \boxed{\frac{573}{20}\text{ km}}$

Cruising time $= \frac{573}{20} \div 3 = \boxed{\frac{573}{60}\text{ min}}$

Total time $= \frac{3}{10} + \frac{3}{5} + \frac{573}{60} = \frac{18+36+573}{60} = \boxed{\frac{627}{60}\text{ min}}$

5. (10 points) A solid pyramid that is 60 feet tall with a square base that is 100 feet wide is to be built out of limestone. (A cubic foot of limestone weighs 175 pounds.)

(a) Set up (but DO NOT EVALUATE) an integral for the **total weight, in pounds**, of the limestone needed to build the pyramid.

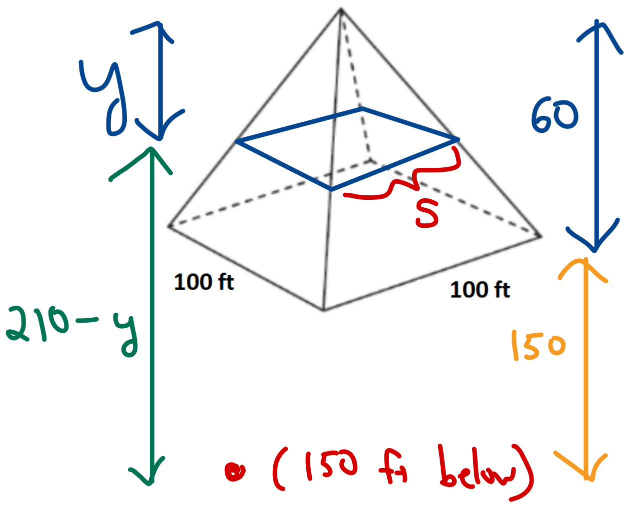
$$\frac{y}{s} = \frac{60}{100}$$

$$s = \frac{10y}{6} = \frac{5}{3}y$$

Area of slice $= \left(\frac{5}{3}y\right)^2$

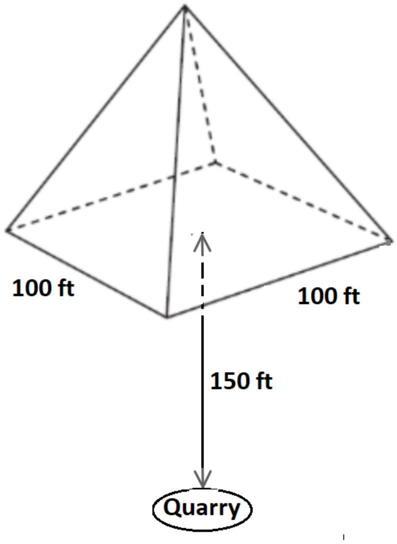
$$\text{Vol} = \int_0^{60} \left(\frac{5}{3}y\right)^2 dy$$

$$\text{Weight} = \boxed{175 \int_0^{60} \left(\frac{5}{3}y\right)^2 dy}$$



(b) Set up (but DO NOT EVALUATE) an integral for the **work, in foot-pounds**, needed to raise the limestone to the required height to build the pyramid, assuming all limestone starts in a quarry that is 150 feet below the base of the pyramid.

$$\boxed{175 \int_0^{60} \left(\frac{5}{3}y\right)^2 (210-y)\, dy}$$



6. (10 points) The region in the first quadrant to the right of the $y$-axis, above the curve $y = 5x^2$, and below the curve $y = x^2 + 1$ is rotated around the $y$-axis to form a solid of revolution.

   (a) Set up a definite integral for the volume of this solid using the method of cylindrical shells, and evaluate the integral to compute the volume.

   (b) Using the washers/disks method, write the volume of this solid in terms of definite integrals. DO NOT EVALUATE the integrals.

7. (a) (4 points) Set up a definite integral for the arclength of the curve

$$y = \sin(2x) \text{ for } 0 \le x \le \pi/4.$$

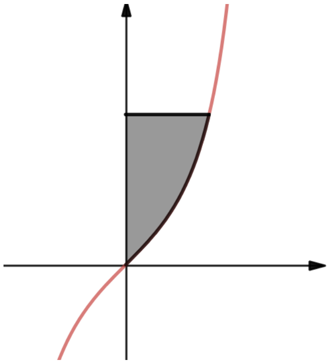
DO NOT EVALUATE THIS INTEGRAL.

(b) (6 points) Approximate the integral in part (a) using the Trapezoid Rule with $n = 3$ subintervals. Simplify your answer, but leave it in exact form, as an expression in terms of $\pi$ and square roots, evaluating all trig functions.

8. (10 points) The region shown below is bounded by the curves

$$y = \sec(x)\tan(x) \text{ and } y = \sqrt{2}, \text{ for } 0 \leq x \leq \pi/4.$$

Compute the $y$-**coordinate**, $\bar{y}$, for the center of mass of a thin lamina of uniform density occupying this region. *Show all work and box your answer. Give your answer in decimal form, rounded to 2 decimal digits.*



9. (10 points) Solve the following differential equation, subject to the stated initial value. Show all steps, and give your answer in explicit form, $y = f(x)$.

$$(x^2 - 2x)y' = (x-4)y, \quad y(1) = -3$$

10. A large vat contains 10 liters of brine (salt dissolved in water). More brine is pumped into the vat at a rate of 2 liters per hour. The incoming brine solution contains 3 grams of salt per liter. The solution in the vat is kept thoroughly mixed and is drained from the vat at a rate of 2 liters per hour.

    (a) (3 points) Set up a **differential equation** for the amount $y = y(t)$ of grams of salt in the vat at $t$ hours. Do not solve yet.

    (b) (5 points) Denote by $y_0$ the intial amount of salt in the vat, in grams, at time $t = 0$ hours. Solve the differential equation to find $y(t)$. Your answer will include the unknown constant $y_0$.

    (c) (2 points) Suppose that after 4 hours, the concentration of the salt in the vat is 4 grams per liter. What was the initial concentration of the salt in the vat (in grams per liter)?