Math 125D 3/1/24

Chapters 9.3 and 9.4

(and 3.8, sorta)

DFEP #15 Solution:

We want to solve $\dfrac{dy}{dx} = xy\sin(x)$ with $y(\pi) = 1$. Separate variables to get

$$\frac{dy}{y} = x\sin(x)\,dx.$$

Integrate (using integration by parts on the right side) to get

$$\ln|y| = -x\cos(x) + \sin(x) + C$$

Plug in $x = \pi$, $y = 1$ to get $0 = \pi + C$, so $C = -\pi$, and we have the equation

$$\ln|y| = -x\cos(x) + \sin(x) - \pi$$

We want the continuous piece of this curve containing $(\pi, 1)$, so $y > 0$ and we have

$$y = e^{-x\cos(x)+\sin(x)-\pi}.$$

DFEP #16: Friday, March 1st.

Westley has a jug that contains 4 liters of wine with 70 grams of Iocaine powder mixed in. At time $t = 0$, he begins pouring in more wine and Iocaine powder at a rate of 0.5 liters of wine and 2 grams of powder per minute. At the same time, the jug is mixed well and 0.5 liters of the mixture are poured out per minute (and safely disposed of).

1 gram of Iocaine powder is fatal to the average person, but Westley can withstand twice that amount. How long should Westley keep mixing the wine in this way so that he can safely drink a 200 mL glass, but no one else can?

Express your answer as an interval of time, e.g.: "He should stop between 5 and 7 minutes".

Ex] Solve the diff eq. $y' = xy^3$ with init. cond. $y(2) = -1$.

$$\frac{dy}{dx} = xy^3 \rightarrow \int \frac{dy}{y^3} = \int x\,dx \rightarrow \frac{-1}{2y^2} = \frac{1}{2}x^2 + C \longrightarrow \frac{1}{y^2} = -x^2 \underbrace{-2C}_{D}$$

$$\frac{1}{y^2} = -x^2 + D \xrightarrow[y=-1]{x=2} 1 = -4 + D \rightarrow D = 5$$

$$\frac{1}{y^2} = -x^2 + 5$$

$$y^2 = \frac{1}{-x^2+5}$$

$$y = \pm\sqrt{\frac{1}{5-x^2}}$$

$y=-1$ is negative

$$\boxed{y = -\sqrt{\frac{1}{5-x^2}}}$$

Chapter 9.4 | Exponential Growth, Newton's Law of Cooling, and the Logistic Curve

covered in 3.8

A common type of diff. eq. is $y' = \overset{\text{constant}}{k}y$.

$$\int \frac{dy}{y} = \int k\,dx$$

$$\ln|y| = kx + C$$

$$|y| = e^{kx+C} = e^{kx}e^{C}$$

$$y = \underbrace{\pm e^{C}}_{A} e^{kx}$$

any real #

Solutions are of the form $y = Ae^{kx}$

or

$y = Ab^{x}$

positive

Or: $e^{kx} = (\underbrace{e^{k}}_{b})^{x} = b^{x}$

How to solve problems like this:

Given: You know the quantity has a diff eq. like this, and you know two points on the curve.

Plug in those points to find A and k.

(or b)

Newton's Law of Cooling: An object with temp T in an environment w/ temp T_s has its temp. described by $\frac{dT}{dt} = k(T - T_s)$ $\longrightarrow$ Solution: $T = T_s + Ae^{kt}$

constant

Ex] A green bean casserole w/ shallots and homemade cream of mushroom soup is removed from a 375°F oven. 10 minutes later, temp. is 250°F, and 20 min later, 175°F. What is the temp. of the room?

$$T = \underbrace{T_s}_{?} + \underbrace{A}_{?} e^{\overset{?}{kt}}$$

$$375 = T_s + A$$

$$250 = T_s + Ae^{10k}$$

$$175 = T_s + Ae^{20k}$$

$$125 = A(1 - e^{10k})$$

$$75 = A(e^{10k} - e^{20k})$$

$$\frac{125}{75} = \frac{1 - e^{10k}}{e^{10k} - e^{20k}}$$

$$\frac{5}{3} = \frac{\cancel{1 - e^{10k}}}{e^{10k}\cancel{(1 - e^{10k})}}$$

$$e^{10k} = \frac{3}{5}$$

$$125 = A\left(\frac{2}{5}\right) \rightarrow A = 312.5$$

$$375 = T_s + 312.5$$

$$T_s = \boxed{62.5° \text{ F}}$$

The Logistic Curve: Sometimes we'll see equations of the form $y' = \underbrace{k}_{\text{constants}} y(\underbrace{L}_{} - y)$

Ex] The population in some city follows the diff eq. $\dfrac{dP}{dt} = kP(1000-P)$. Find a general solution for P in terms of t.

$$\int \frac{dP}{P(1000-P)} = \int k\,dt \longrightarrow \frac{1}{1000}\int\left(\frac{1}{P} + \frac{1}{1000-P}\right)dP = \int k\,dt$$

$$\frac{1}{1000}\Big(\ln P - \ln(1000-P)\Big) = kt + C$$

$$\ln(P) - \ln(1000-P) = 1000kt + \overbrace{1000C}^{D}$$

$$\ln\left(\frac{P}{1000-P}\right) = 1000kt + D$$

$$\frac{P - 1000 + 1000}{1000-P} = e^{1000kt}\underbrace{e^{D}}_{A}$$

$$-1 + \frac{1000}{1000-P} = Ae^{1000kt}$$

$$\frac{1000}{1000-P} = 1 + Ae^{1000kt}$$

$$\frac{1000-P}{1000} = \frac{1}{1+Ae^{kt}}$$

$$1000 - P = \frac{1000}{1+Ae^{1000kt}}$$

$$\boxed{P = 1000\left(1 - \frac{1}{1+Ae^{1000kt}}\right)}$$

Partial frac:

$$\frac{1}{P(1000-P)} = \frac{A}{P} + \frac{B}{1000-P}$$

$$1 = A(1000-P) + BP$$

$\xrightarrow{1000}$ $1 = 1000B \qquad B = \frac{1}{1000}$

$1 = 1000A \qquad A = \frac{1}{1000}$

Ex] The population in some city follows the diff eq. $\frac{dP}{dt} = kP(1000-P)$.

a) Find a general solution for P in terms of t.

$$\boxed{P = 1000\left(1 - \frac{1}{1+Ae^{1000kt}}\right)}$$

b) Suppose the current pop is 250, and one year from now it's 400

When will pop = 600?

$$250 = 1000\left(1 - \frac{1}{1+A}\right) \rightarrow \frac{1}{4} = 1 - \frac{1}{1+A} \rightarrow \frac{1}{1+A} = \frac{3}{4} \rightarrow 1+A = \frac{4}{3} \quad A = \frac{1}{3}$$

$$400 = 1000\left(1 - \frac{1}{1+Ae^{1000k}}\right) \rightarrow \frac{2}{5} = 1 - \frac{1}{1+\frac{1}{3}e^{1000k}} \rightarrow \frac{1}{1+\frac{1}{3}e^{1000k}} = \frac{3}{5}$$

$$1 + \frac{1}{3}e^{1000k} = \frac{5}{3} \quad \frac{1}{3}e^{1000k} = \frac{2}{3}$$

$$e^{1000k} = 2$$

$$P = 1000\left(1 - \frac{1}{1+\frac{1}{3}(2^t)}\right)$$

$$600 = 1000(\text{etc...}) \rightarrow \boxed{P \approx 2.17 \text{ years}}$$