Math 125D 2/28/24

Chapter 9.3

DFEP #14 Solution:

We want the centroid of the region bounded by $y = e^x$, $y = \sin(x)$, $x = 0$, and $x = \pi$, and to do that we'll need three things: the total mass m, and the moments M_y and M_x. Note that $e^x > \sin(x)$ for this whole region.

$$m = \int_0^\pi \rho(e^x - \sin(x))\,dx = \rho\left(e^x + \cos(x)\right)\Big]_0^\pi = \rho(e^\pi - 3)$$

$$M_y = \int_0^\pi \rho x(e^x - \sin(x))\,dx = \rho x(e^x + \cos(x))\Big]_0^\pi - \int_0^\pi \rho(e^x + \cos(x))\,dx$$

$$= \rho\left(xe^x + x\cos(x) - e^x - \sin(x)\right)\Big]_0^\pi = \rho(\pi e^\pi - \pi - e^\pi + 1)$$

$$M_x = \int_0^\pi \frac{1}{2}\rho\left(e^{2x} - \sin^2(x)\right)\,dx = \rho\left(\frac{1}{4}e^{2x} - \frac{1}{4}x + \frac{1}{8}\sin(2x)\right)\Big]_0^\pi$$

$$M_x = \rho(\frac{1}{4}e^{2\pi} - \frac{\pi}{4} - \frac{1}{4})$$

So the centroid is at $\left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{\pi e^\pi - \pi - e^\pi + 1}{e^\pi - 3}, \dfrac{e^{2\pi} - \pi - 1}{4e^\pi - 12}\right)$.

DFEP #15: Wednesday, February 28th:

Solve the differential equation $y' = xy\sin(x)$ with initial condition $y(\pi) = 1$.

New rule for the final: do not leave answers aggressively unsimplified.

Example:

Great: $5e^4 - e^2$

Sure: $e^2(5e^2 - 1)$

Okay: $\frac{1}{2}(10e^4 - 2e^2)$

Okay, but try to avoid: ≈ 265.602

Not okay: $\frac{1}{2}(16e^4 - 4e^2 - (8e^4 - 4e^2 - 2(e^4 - e^2)))$

Also not okay: $\frac{1}{2}(4^2e^4 - 2(4)e^4 + 2e^4) - \frac{1}{2}(2^2e^2 - 2(2)e^2 + 2e^2)$

Chapter 9.3: Separable Differential Equations

A first-order diff. eq. is "separable" if it can be written as $y' = f(x)g(y)$

any functions

y', but not y'', y''', etc

Ex) These are separable:

$y' = xy^2 \qquad y' = 7y^3 \qquad y' = \sin(x)e^y \qquad y' - 2xy' = \sin(y)$

how?

$y'(1-2x) = \sin(y)$

$y' = \left(\frac{1}{1-2x}\right)\sin(y)$

How to solve them: 1) Write y' as $\frac{dy}{dx}$.

2) Move all 'x's to one side, and all 'y's to the other. (Including dx & dy)

3) You now have $\_\_\,dy = \_\_\,dx$. Antidiff. both sides. $\int \_\_\,dy = \int \_\_\,dx$.

4) Solve for y in the resulting equation.

? 5) If there's an initial condition, plug in in, solve for unknown constants.

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Ex] a) Find a general solution to the diff. eq $y' = x^2 y$.

$$\frac{dy}{dx} = x^2 y \rightarrow \frac{dy}{y} = x^2 dx \rightarrow \int \frac{dy}{y} = \int x^2 dx \rightarrow \ln|y| = \frac{1}{3}x^3 + C \xrightarrow{e^{\sim}} e^{\ln|y|} = e^{\frac{1}{3}x^3 + C}$$

$$|y| = e^{\frac{1}{3}x^3} e^{C} \rightarrow y = \pm e^{\frac{1}{3}x^3} e^{C} \rightarrow \boxed{y = \underbrace{D}_{\text{D is some constant}} e^{\frac{1}{3}x^3}}$$

$\pm e^{C} = D$ another constant

b) Find a sol'n w/ init. cond. $\underbrace{y(3) = 7}_{x=3,\ y=7}$.

$$7 = De^{\frac{1}{3}(3^3)} \qquad 7 = De^{9} \qquad D = \frac{7}{e^{9}}$$

$$\boxed{y = \frac{7}{e^{9}} e^{\frac{1}{3}x^3}}$$

Ex) Solve $y' = xy^2 + x + y^2 + 1$ w/ init. cond. $y(2) = 1$

$y' = x(y^2+1) + y^2+1 \rightarrow y' = (x+1)(y^2+1) \rightarrow \frac{dy}{y^2+1} = (x+1)\,dx \rightarrow \int \frac{1}{y^2+1}\,dy = \int (x+1)\,dx$

$\rightarrow \arctan(y) = \frac{1}{2}x^2 + x + C \rightarrow \arctan(y) = \frac{1}{2}x^2 + x + \frac{\pi}{4} - 4$

$x=2,\ y=1$

$\arctan(1) = 2 + 2 + C$

$\frac{\pi}{4} = 4 + C \rightarrow C = \frac{\pi}{4} - 4$

$$\boxed{y = \tan\left(\frac{1}{2}x^2 + x + \frac{\pi}{4} - 4\right)}$$

Ex)

A 20-gallon tank contains 5 gallons of lemonade and 15 gallons of iced tea.
A 50-50 mix of iced tea & lemonade is poured into the tank at a rate of 2 gallons per minute. The tank is mixed constantly, and 2 gallons of the mixture is poured out per minute.
After 30 minutes, how much lemonade is in the tank?

Need to turn this into a differential equation!

L = amount of lemonade

t = time (in minutes)

Init. cond.: $L(0)=5$

$t=0,\ L=5$

what fraction of the tank is lemonade = $\dfrac{\text{amount of lemonade}}{\text{total volume}}$

$$\frac{dL}{dt} = \underbrace{1}_{\text{rate in}} - \underbrace{\left(\frac{L}{20}\right)2}_{\text{rate out}}$$

$$\frac{dL}{dt} = 1 - \frac{L}{10} = \frac{10-L}{10}$$

$$\int \frac{dL}{10-L} = \int \frac{dt}{10}$$

$$-\ln|10-L| = \frac{t}{10} + C$$

$$|10-L| = e^{\frac{-t}{10}} e^{-C}$$

$$10-L = \underbrace{\pm e^{-C}}_{D} e^{-t/10}$$

$$L = 10 - De^{-t/10}$$

$5 = 10 - D \qquad D=5$

$$L = 10 - 5e^{\frac{-t}{10}}$$

$$L = \boxed{10 - 5e^{-3} \text{ gallons}}$$

More examples next time