Math 125D 2/27/24

Chapters 8.3 & 9.1

DFEP #14: Monday, February 26th.

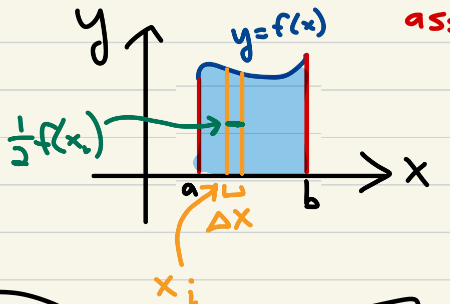
Compute the centroid of the region bounded by the curves $y = e^x$, $y = \sin(x)$, $x = 0$, and $x = \pi$.

Chapter 8.3, Continued

Last time: $M_y = \sum(\text{mass}\cdot x\text{-coord})$, $M_x = \sum(\text{mass}\cdot y\text{-coord})$, C.o.M $= (\bar{x}, \bar{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right)$

total mass

Want center of mass of:



assume $\overbrace{\text{uniform density}}^{\text{in Math 126, not uniform}}$ ρ. $\rho \to \dfrac{\text{mass}}{\text{area}}$

Cut into pieces:

$$M_y = \sum\Big(\overbrace{(\text{mass of strip})}^{\rho\cdot\overbrace{\Delta x f(x_i)}^{\text{area}}}\overbrace{(x\text{-coord})}^{x_i}\Big) \xrightarrow{\text{take lim}} \int_a^b \rho x f(x)\,dx = M_y$$

$$M_x = \sum\Big(\overbrace{(\text{mass of strip})}^{\rho\Delta x f(x_i)}\overbrace{(\text{avg. } y\text{-coord})}^{\frac{1}{2}f(x_i)}\Big) \xrightarrow{\text{take lim}} \int_a^b \frac{1}{2}\rho\big(f(x)\big)^2\,dx = M_x$$

$$\text{mass} = \sum(\text{mass of strip}) \xrightarrow{\text{lim}} \int_a^b \rho f(x)\,dx = m$$

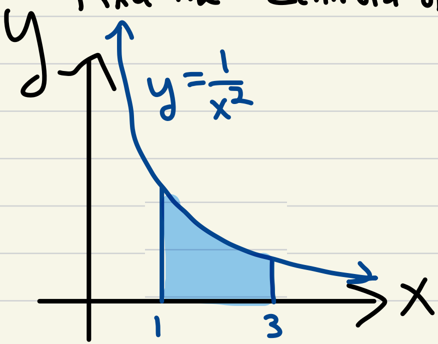
<u>Centroid</u>: Center of mass of this shape, assuming uniform density.

$$\text{C.o.M.} = (\bar{x}, \bar{y}) = \left(\frac{\int_a^b \cancel{\rho} x f(x)\,dx}{\int_a^b \cancel{\rho} f(x)\,dx}, \frac{\int_a^b \frac{1}{2}\cancel{\rho}(f(x))^2\,dx}{\int_a^b \cancel{\rho} f(x)\,dx}\right)$$

(If ρ is constant, it cancels! Can ignore ρ in this class)

Ex) Let R be the region bounded by $y=0$, $y=\frac{1}{x^2}$, $x=1$, and $x=3$.

Find the centroid of R.



$$M_y = \int_1^3 \frac{x}{x^2}\,dx = \int_1^3 \frac{1}{x}\,dx = \ln|x|\Big]_1^3$$

$$= \ln(3)$$

$$M_x = \int_1^3 \frac{1}{2}\left(\frac{1}{x^2}\right)^2 dx = \frac{1}{2}\int_1^3 \frac{1}{x^4}\,dx = \frac{-1}{6}\frac{1}{x^3}\Big]_1^3$$

$$= \frac{-1}{6}\left(\frac{1}{27} - 1\right) = \frac{13}{81}$$

$$m = \int_1^3 \frac{1}{x^2}\,dx = \left(\frac{-1}{x}\right)\Big]_1^3 = \frac{-1}{3} + 1 = \frac{2}{3}$$

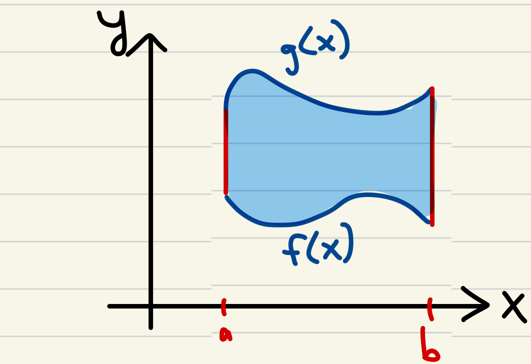
$$\text{centroid} = \left(\frac{\ln(3)}{\frac{2}{3}}, \frac{13/81}{\frac{2}{3}}\right) = \boxed{\left(\frac{3\ln(3)}{2}, \frac{13}{54}\right)}$$

$$\int_a^b \rho\, x f(x)\,dx = M_y$$

$$\int_a^b \frac{1}{2}\rho\left(f(x)\right)^2 dx = M_x$$

$$\int_a^b \rho f(x)\,dx = m$$

What if the lower bound isn't $y=0$?



$$m = \int_a^b \overbrace{\rho\big(g(x)-f(x)\big)}^{\text{mass of vert. strip}}\,dx$$

$$M_y = \int_a^b \rho\, x\big(g(x)-f(x)\big)\,dx$$

$$M_x = \int_a^b \rho\big(g(x)-f(x)\big)\overbrace{\left(\tfrac{1}{2}\big(g(x)+f(x)\big)\right)}^{\text{avg y-coord.}}\,dx$$

simplifies

$$= \int_a^b \tfrac{1}{2}\rho\left(\big(g(x)\big)^2-\big(f(x)\big)^2\right)dx$$

Chapter 9.1: Differential Equations

A differential equation is an equation relating x and/or y to one or more derivatives y', y'', etc.

Ex) $\underbrace{y' = y^2 + 3x}_{\text{1st order}}$ $\quad$ $\underbrace{y' + y'' = 2xy}_{\text{2nd order}}$

The order of a diff. eq. is the highest derivative that it uses.

A solution to a diff eq is an equation only using x & y (no deriv.) which satisfies that diff. eq:

Ex) Is $y = \frac{2}{3}e^{x} + e^{-2x}$ a solution to the diff. eq. $y' + 2y = 2e^{x}$?

$y' = \frac{2}{3}e^{x} - 2e^{-2x}$ → Plug in: does it work?

$$\left(\frac{2}{3}e^{x} - 2e^{-2x}\right) + 2\left(\frac{2}{3}e^{x} + e^{-2x}\right) \stackrel{?}{=} 2e^{x}$$

$\boxed{\text{Yes!}}$

Ex] Find all constant solutions to the diff eq. $y' - 3y'' + (y''')^7 = y^3 - 4y^2 - 21y$

$y = C$

constant

So y', y'', y''' zero.

$0 = y^3 - 4y^2 - 21y$

$0 = y(y^2 - 4y - 21)$

$0 = y(y-7)(y+3)$

$\boxed{y = 0,\ y = 7,\ \text{and}\ y = -3}$

Ex] Is $y = \underbrace{A}_{\text{constant}}\cos(2x)$ a solution to $y'' = -4y$?

$y' = -2A\sin(2x)$

$y'' = -4A\cos(2x)$

$\boxed{\text{Yes!}}$

Ex] Find any solution to $y'' = -4y$ with the initial condition $\underbrace{y(\pi) = 6}_{\text{when } x = \pi,\ y = 6}$. Can use $y = A\cos(2x)$

$6 = A\underbrace{\cos(2\pi)}_{1}$

Let $A = 6$: $\boxed{y = 6\cos(2x)}$