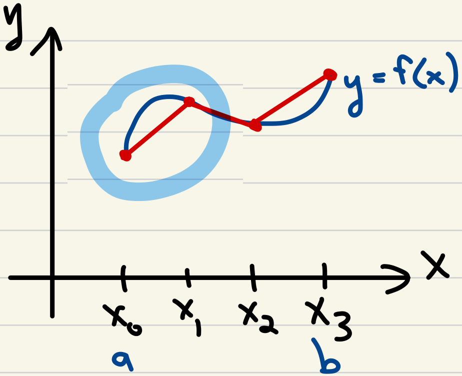
Math 125D 2/23/24

Chapters 8.1 & 8.3

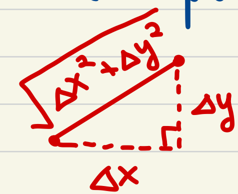
Chapter 8.1: Arc Length

how can we find the length of a curve $y=f(x)$?



Idea:
- Break into pieces
- Approximate each piece (as a line)
- Let # pieces → ∞

Length of each piece:



$$\underbrace{\frac{\Delta y}{\Delta x}}_{\text{slope of secant line}} \approx \underbrace{f'(x_i)}_{\text{slope of tan. line}} \rightarrow \boxed{\Delta y \approx \Delta x f'(x)}$$

$$\text{Length of each piece} = \sqrt{\Delta x^2+\Delta y^2} = \sqrt{\Delta x^2+(\Delta x f'(x_i))^2} = \sqrt{1+(f'(x_i))^2}\,\Delta x$$

↓ Σ, take limit

$$\boxed{\int_a^b \sqrt{1+(f'(x))^2}\,dx} \leftarrow \text{length of } y=f(x) \text{ from } x=a \text{ to } x=b.$$

$$\boxed{\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx} \leftarrow \text{length of } y=f(x) \text{ from } x=a \text{ to } x=b.$$

Ex) Find the length of $\underbrace{y=\sqrt{x^3}}_{y'=\frac{3}{2}\sqrt{x}}$ from $x=0$ to $x=16$

$$\int_0^{16} \sqrt{1+\left(\tfrac{3}{2}\sqrt{x}\right)^2}\,dx$$

$$= \int_0^{16} \sqrt{1+\tfrac{9}{4}x}\,dx = \frac{4}{9}\int_1^{37} \sqrt{u}\,du = \frac{4}{9}\left(\frac{2}{3}u^{3/2}\right)\Big]_1^{37} = \boxed{\frac{8}{27}\left(37^{3/2}-1\right)}$$

$u = 1+\frac{9}{4}x$

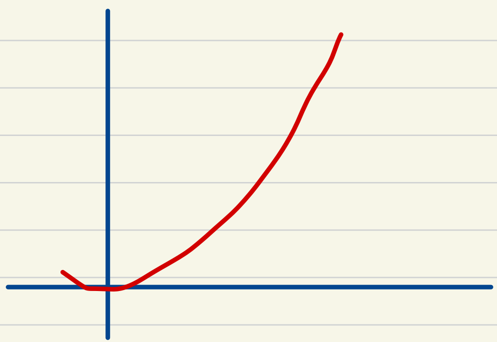
$du = \frac{9}{4}dx$

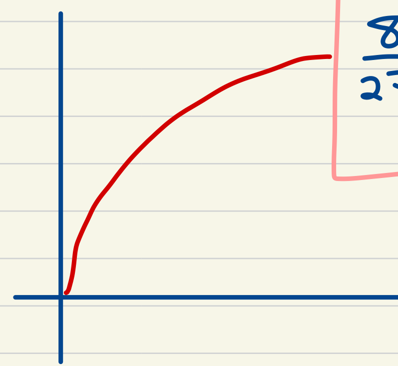
Ex] Find the length of $y=\sqrt[3]{x^2}$ from $x=0$ to $x=8$.

$$y' = \frac{2}{3\sqrt[3]{x}}$$

$$\int_0^8 \sqrt{1+\frac{4}{9\sqrt[3]{x^2}}}\,dx$$ Yuck!

$x=\sqrt{y^3}$ which is the previous problem





$$\boxed{\frac{8}{27}\left(37^{3/2}-1\right)}$$

Ex] Find the length of $y = e^{-x}$ from $x = \frac{-\ln(8)}{2}$ to $x = \frac{-\ln(3)}{2}$.

$$y' = -e^{-x}$$

$$(y')^2 = \left(e^{-x}\right)^2 = e^{-2x}$$

$$\int_{\frac{-\ln(8)}{2}}^{\frac{-\ln(3)}{2}} \sqrt{1+e^{-2x}}\, dx$$

$$u = \sqrt{1+e^{-2x}}$$

$$du = \frac{-2e^{-2x}}{2\sqrt{1+e^{-2x}}}\, dx$$

$$\sqrt{1+e^{-2\left(\frac{-\ln(8)}{2}\right)}} = \sqrt{1+8} = 3$$

$$u\, du = -e^{-2x}\, dx \rightarrow u\, du = (1-u^2)\, dx$$

$$u^2 = 1+e^{-2x}$$

$$-e^{-2x} = 1-u^2$$

$$dx = \frac{u}{1-u^2}\, du$$

$$\int_3^2 \frac{u^2}{1-u^2}\, du = \int_2^3 \frac{u^2-1+1}{u^2-1}\, du = \int_2^3 \left(1 + \frac{1}{u^2-1}\right) du$$

$$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$1 = A(u+1) + B(u-1)$$

-1: $1 = -2B$, $B = \frac{-1}{2}$

$+1$: $1 = 2A$, $A = \frac{1}{2}$

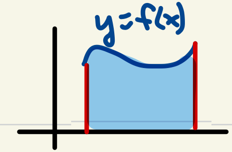
$$= \int_2^3 \left(1 + \frac{1/2}{u-1} - \frac{1/2}{u+1}\right) du$$

$$= \left(u + \tfrac{1}{2}\ln|u-1| - \tfrac{1}{2}\ln|u+1|\right)\Big]_2^3$$

$$= 1 + \tfrac{1}{2}\Big(\ln 2 - \ln 4 - \underbrace{\ln 1}_{0} + \ln 3\Big) = \boxed{1 + \tfrac{1}{2}\ln\left(\tfrac{3}{2}\right)}$$

Chapter 8.3: Center of Mass

Goal: find C.o.M. of

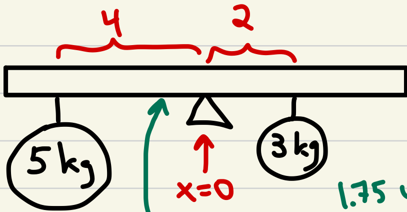


(next time)

1-D version:

Currently lopsided.

Where would this scale be balanced?



$$\text{Moment} = \sum(\text{mass} \cdot \text{position})$$

$$= 5(-4) + 3(2) = -14$$

1.75 units left

$$\text{Center of mass} = \frac{\text{moment}}{\underbrace{\text{total mass}}_{\text{"average" of mass's location}}} = \frac{-14}{8} = \boxed{-1.75}$$

$$M_y = (5\cdot(-2)) + (3\cdot 3) + (2\cdot 1)$$

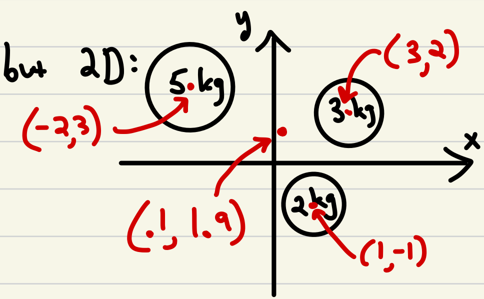
$$= 1$$

$$M_x = (5\cdot 3) + (3\cdot 2) + (2\cdot(-1)) = 19$$

$$m = 10$$

$$(\bar{x},\bar{y}) = \left(\frac{1}{10}, \frac{19}{10}\right) = \boxed{(0.1, 1.9)}$$

Same idea, but 2D:



M_y = "moment about y-axis" = $\sum(\text{mass} \cdot x\text{-coord})$

M_x = "moment about x-axis" = $\sum(\text{mass} \cdot y\text{-coord})$

m = total mass

$$\text{Center of mass} = (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$