Math 125D 2/16/24

Chapter 7.8

DFEP #12 Solution:

So we want to estimate $\frac{1}{12}\int_{-1}^{11}\sqrt{x^3+5}\,dx$ using Simpson's Rule with $n=6$. That means $\Delta x = 2$, so we have:

$$\frac{1}{12}\cdot\frac{2}{3}\left(\sqrt{4}+4\sqrt{6}+2\sqrt{32}+4\sqrt{130}+2\sqrt{348}+4\sqrt{734}+\sqrt{1336}\right)$$

DFEP #13: Friday, February 16th.

Determine whether the following integral is convergent or divergent. (You do not need to evaluate the integral.)

$$\int_0^\infty \frac{dx}{\sqrt{x^3+5}}$$

A List of Topics for the Second Midterm

Here's what you should be able to do for the midterm next week.

Old Stuff

1. Riemann sums
   (a) Compute L_n, R_n, and M_n estimates for areas under curves.
   (b) Write the (exact) area under a curve as a limit of Riemann sums and (for certain curves) evaluate that limit.
   (c) Recognize such a limit, convert it to an integral, and compute it.
2. Integration
   (a) Find antiderivatives of certain elementary functions including polynomials, exponential functions, and certain trigonometric functions.
   (b) Use u-substitution to evaluate more challenging integrals.
   (c) Compute indefinite integrals and definite integrals.
   (d) Evaluate integrals of odd or even functions on intervals of the form $[-a, a]$.
   (e) Use the fundamental theorem of calculus to differentiate functions that are defined in terms of integrals.
3. Applications
   (a) Given velocity or acceleration, compute the net displacement of an object over a time interval *or* compute its total distance traveled.
   (b) Find the area bounded by two or more curves in the plane.

New Stuff

4. More applications
   (a) Compute the volumes of solids by integrating their cross-sectional areas.
   (b) In particular, use the washer method for finding volumes of solids of revolution by integrating along the axis of rotation.
   (c) Also, find volumes of solids of revolution using the shell method.
   (d) Compute the work required to perform certain tasks.
   (e) Find the average value of a function over an interval.
5. More integration techniques
   (a) Understand how to use trigonometric identities to compute integrals of the forms $\int \sin^m(x)\cos^n(x)\,dx$ or $\int \tan^m(x)\sec^n(x)\,dx$.
   (b) Know how and when to use the following techniques:
      - Integration by parts
      - Trigonometric substitution
      - Integration with partial fractions

6. Integral approximation
   (a) Approximate integrals with the trapezoid rule or Simpson's rule.
   (b) Know when L_n, R_n, M_n, or T_n are underestimates or overestimates.
7. Improper integrals
   (a) Evaluate type-1 and type-2 improper integrals.
   (b) Use integral comparison to tell whether certain integrals converge or diverge, even when their integrands are hard to antidifferentiate explicitly.

Chapter 7.8, Continued

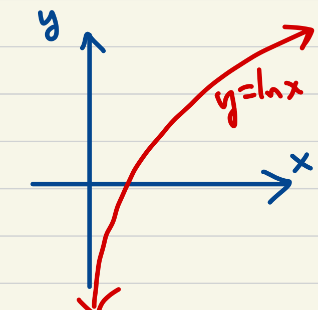
Last time, learned about "Type I" improper integrals: where some bound is $\pm\infty$.

To compute: $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^{t} f(x)\,dx.$

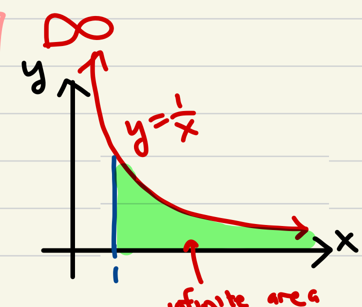
This is a limit! It may or may not exist. If it does exist and is a real number, we say the integral "converges", otherwise we say it "diverges".

Ex) Evaluate $\displaystyle\int_1^{\infty}\frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^{t}\frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t = \lim_{t\to\infty}\Big(\ln(t) - \underbrace{\ln(1)}_{0}\Big) = \infty$

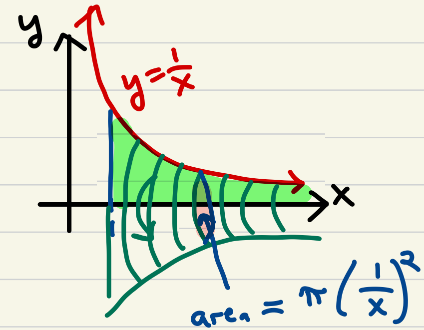
$\ln(t) \to \infty$



$\boxed{\text{So the integral diverges}}$



Ex) Find the volume of the solid formed by rev. the area under $y = \frac{1}{x}$ & above $y=0$ from $x=1$ to ∞ around the x-axis.



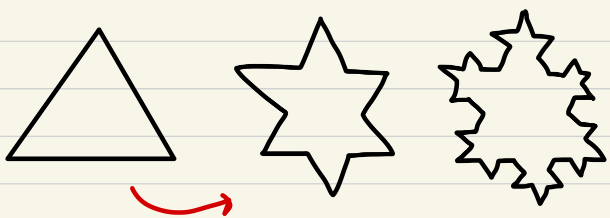
Disc method:

$$\int_1^\infty \pi \frac{1}{x^2}\,dx = \pi \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \pi \lim_{t\to\infty} \left(-\frac{1}{x} \Big]_1^t \right)$$

$$= \pi \lim_{t\to\infty} \left(\underbrace{\frac{-1}{t}}_{\to 0} + 1 \right) = \boxed{\pi}$$

It converges!

Dimensions are weird!



→ finite area but inf. perimeter

When does $\int_a^\infty \frac{1}{x^p}\,dx$ converge?

$a>0$ is constant

p is an arbitrary constant

When $p=1$, diverges

$$\lim_{t\to\infty}\int_a^t \frac{1}{\underbrace{x^p}_{x^{-p}}}\,dx = \lim_{t\to\infty}\left(\left.\frac{x^{-p+1}}{-p+1}\right]_a^t\right)$$

$$=\lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \underbrace{\frac{a^{-p+1}}{-p+1}}_{\text{constant}}\right)$$

converges if $-p+1<0$

Only converges if $p>1$

"p-test"

Ex) Do these converge or not?

a) $\int_1^\infty \frac{1}{x^7}\,dx$ $\boxed{\text{Converges,}}$ b/c $7>1$

b) $\int_4^\infty \frac{1}{x^{0.3}}\,dx$ $\boxed{\text{Diverges}}$

c) $\int_{-3}^\infty \frac{1}{x^4}\,dx$

This has a vertical asymptote inside the bounds!

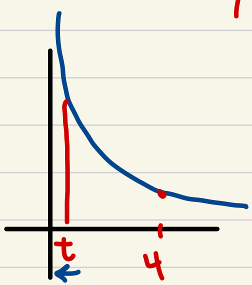
Can't use p-test.

This is a "Type II" improper integral.

Type II improper integrals: When there is a vertical asymptote at one or both bounds, or within the domain of integration. Again, use limits to approach the "illegal" bound.

Ex] Find $\displaystyle\int_0^4 \underbrace{\frac{1}{\sqrt{x}}}_{\text{asymptote @ } x=0} dx = \lim_{t\to 0^+}\int_t^4 \frac{1}{\sqrt{x}}dx$

$$= \lim_{t\to 0^+}\left(2\sqrt{x}\Big]_t^4\right) = \lim_{t\to 0^+}\left(4-\underbrace{2\sqrt{t}}_{\to 0}\right) = \boxed{4}$$

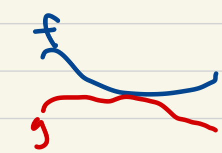


Ex] $\displaystyle\int_0^4 \frac{1}{x^2}dx = \lim_{t\to 0^+}\int_t^4 \frac{1}{x^2}dx = \lim_{t\to 0^+}\left(\frac{-1}{x}\Big]_t^4\right) = \lim_{t\to 0^+}\left(\frac{-1}{4}+\underbrace{\frac{1}{t}}_{\to\infty}\right)$

Diverges

Integral comparison:

Say $f(x)\geq g(x)\geq 0$ on some domain.

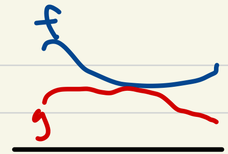


If $\int_a^b f(x)dx$ converges, $\int_a^b g(x)dx$ does too

If $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges too.

Integral comparison:

Say $f(x) \geq g(x) \geq 0$ on some domain.



If $\int_a^b f(x)\,dx$ converges, $\int_a^b g(x)\,dx$ does too

If $\int_a^b g(x)\,dx$ diverges, then $\int_a^b f(x)\,dx$ diverges too.

Ex) Does $\int_1^\infty \frac{\sin^2 x}{x^2+1}\,dx$ converge or diverge?

Let's compare. $0 \leq \sin^2 x \leq 1$

So $0 \leq \frac{\sin^2 x}{x^2+1} \leq \frac{1}{x^2+1}$

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \frac{\pi}{4} \text{ (from Wednesday)}$$

Since the bigger function converges, $\int_1^\infty \frac{\sin^2 x}{x^2+1}\,dx$ does too.

What if multiple "parts" of an integral are improper? Break into multiple pieces.

If *any* of those pieces diverge, so does the whole thing. Otherwise, just add the limits together.

Ex] $$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{x}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty}\left(\int_{t}^{0} \frac{x}{1+x^2}\,dx\right) + \lim_{t\to\infty}\left(\int_{0}^{t} \frac{x}{1+x^2}\,dx\right)$$

etc.