Math 125D 2/14/24

Chapter 7.7

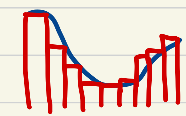
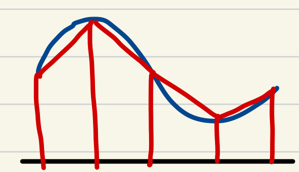
(and 7.8?)

DFEP #12: Wednesday, February 14th.

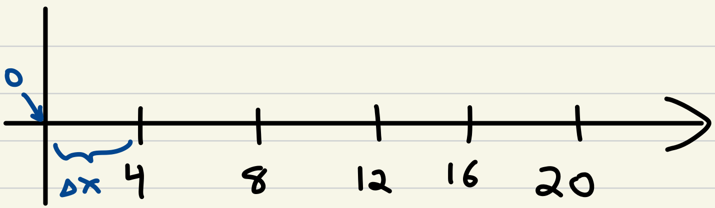
Use Simpson's Rule with $n = 6$ to estimate the average value of $f(x) = \sqrt{x^3 + 5}$ on the interval $[-1, 11]$.

Comparison of approximations: how accurate are they?

don't have to learn this

Approximation	error bound	What's K?	Overestimate when...
R_n	$\frac{K(b-a)}{n}$	bound on $\lvert f'(x)\rvert$	f increasing 
L_n			f decreasing
M_n	$\frac{K(b-a)^3}{24n^2}$	how big $\lvert f''(x)\rvert$ gets	f concave down 
T_n	$\frac{K(b-a)^3}{12n^2}$		f concave up 

Ex] Use the trap. rule to estimate $\int_0^{20} \sin(x)\,dx$ w/ $n=5$ subintervals.



$$T_n = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right)$$

$$\boxed{T_5 = \frac{4}{2}\left(\underset{0}{\cancel{\sin(0)}} + 2\sin(4) + 2\sin(8) + 2\sin(12) + 2\sin(16) + \sin(20)\right)}$$

Simpson's Rule: What's better than connecting $\underbrace{\text{pts w/ straight lines}}_{\text{to make Trapezoids}}$? Use parabolas.

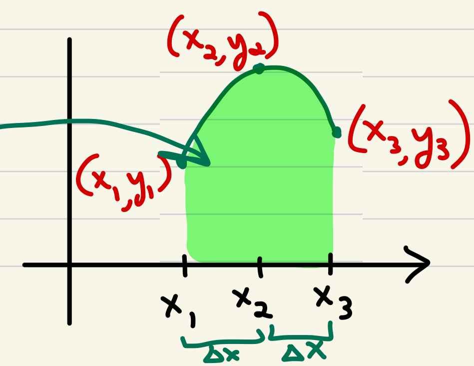
Ex] $n=6$ — must be even

Pair up consec. subint., draw parabolas through the 3 pts in that pair.

Add up areas under parabolas.

$$\text{Area} = \frac{\Delta x}{3}\left(y_1 + 4y_2 + y_3\right)$$

(Why? We'll see later)



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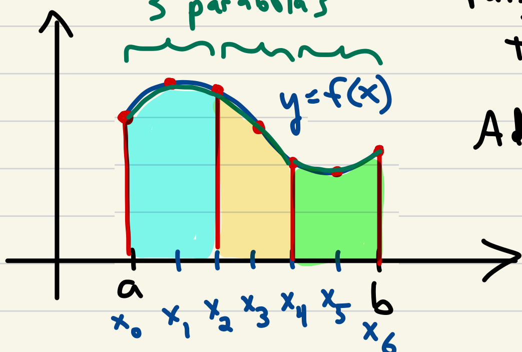

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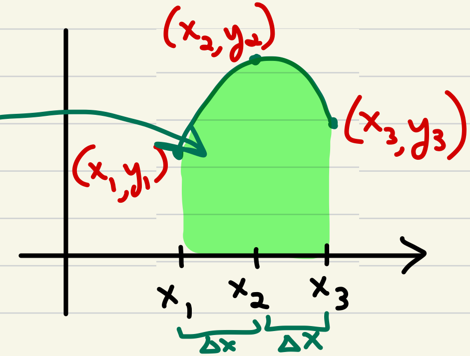


Area under these 3 parabolas:

$$\text{Area} = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \frac{\Delta x}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right) + \frac{\Delta x}{3}\left(f(x_4) + 4f(x_5) + f(x_6)\right)$$

$$= \frac{\Delta x}{3}\left(\underbrace{f(x_0)}_{1} + \underbrace{4f(x_1)}_{4} + \underbrace{2f(x_2)}_{2} + \underbrace{4f(x_3)}_{4} + \underbrace{2f(x_4)}_{2} + \underbrace{4f(x_5)}_{4} + \underbrace{f(x_6)}_{1}\right)$$

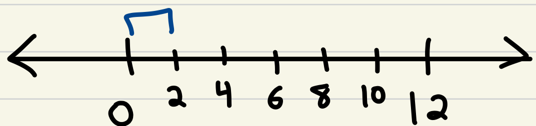
Simpson's Rule: Area is approximately $\frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\dots+4f(x_{n-1})+f(x_n)\right)$

1 4 2 4 2 4 2 4 .. 4 1

error bound:
$$\frac{K(b-a)^5}{180n^4}$$
$$K \ge |f^{(4)}(x)|$$

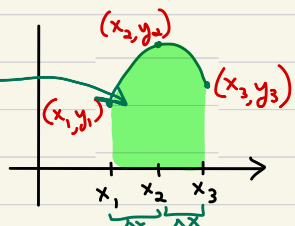
Ex] Approximate $\int_0^{12} \sin(x^2)\,dx$ w/ Simpson's Rule, $n=6$.

$\Delta x = 2$

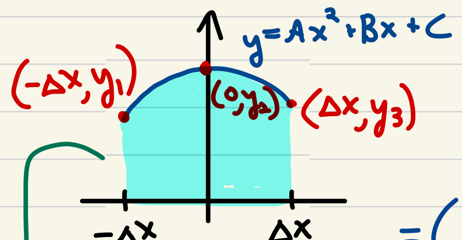


$$\approx \frac{2}{3}\left(\cancel{\sin(0)}+4\sin(4)+2\sin(16)+4\sin(36)+2\sin(64)+4\sin(100)+\sin(144)\right)$$

Why does this work?



Area = $\frac{\Delta x}{3}(y_1+4y_2+y_3)$



$$\text{Area} = \int_{-\Delta x}^{\Delta x}(Ax^2+Bx+C)\,dx$$

algebra

$$= \left(\tfrac{1}{3}Ax^3+\tfrac{1}{2}Bx^2+Cx\right)\Big]_{-\Delta x}^{\Delta x} = \tfrac{2}{3}A\Delta x^3 + 2C\Delta x$$

$y_1 = A\Delta x^2 - B\Delta x + C$

$y_2 = C$

$y_3 = A\Delta x^2 + B\Delta x + C$

$y_1+4y_2+y_3 = 2A\Delta x^2 + 6C$

$\frac{\Delta x}{3}$

$$\frac{\Delta x}{3}(y_1+4y_2+y_3) = \frac{2}{3}A\Delta x^3 + 2C\Delta x$$

Same!

Chapter 7.8 | Improper Integrals

"Improper Type I" integral: $\displaystyle\int_1^{\infty} \frac{1}{1+x^2}\,dx$

uh oh

∞ is not a real number, so we can't plug in into the antiderivative.

Main idea: Consider $\displaystyle\int_1^t \frac{1}{1+x^2}\,dx$, then let $t \to \infty$.

$$\lim_{t\to\infty}\int_1^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\left(\arctan x\Big]_1^t\right) = \lim_{t\to\infty}\left(\arctan t - \underbrace{\arctan 1}_{\frac{\pi}{4}}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$