Math 125D 2/9/24

Chapter 7.4

DFEP #10 Solution:

We want $\displaystyle\int_0^2 \left(\frac{2x^3+10x^2+16x+7}{x^2+5x+6} - 2x \right) dx = \int_0^2 \frac{4x+7}{(x+1)(x+5)}\, dx$. Okay, cool, let's do partial fractions:

$$\begin{aligned}
&\int_0^2 \left(\frac{3/4}{x+1} + \frac{13/4}{x+5} \right) dx \\
&= \left(\frac{3}{4} \ln|x+1| + \frac{13}{4} \ln|x+5| \right) \Bigg]_0^2 \\
&= \frac{3}{4} \ln(3) + \frac{13}{4} \ln(7) - \frac{13}{4} \ln(6)
\end{aligned}$$

DFEP #11: Friday, February 9th.

More integrals!

(a) $\displaystyle\int \frac{4x^4+10x^3+9x^2+16x+8}{x^3+2x^2+x} dx$

(b) $\displaystyle\int \sin^5(x)\cos^8(x)\, dx$

(c) $\displaystyle\int \frac{\sqrt{16+x^2}}{x} dx$

What about when deg. of num < deg. of denom? Try to factor the denominator, and then use "partial fractions" to "undo" fraction addition:

Ex) $\int \frac{3x+8}{x^2-4x}dx = \int \frac{3x+8}{x(x-4)}dx$

Want to split up $\frac{3x+8}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$ Constant

i.e. "splitting up common denominator"

clear denominators

$3x+8 = A(x-4) + Bx$ should be equal for <u>all</u> x

$x=4$: $20 = 4B \rightarrow B=5$

$x=0$: $8 = -4A \rightarrow A=-2$

$\int \left(\frac{-2}{x} + \frac{5}{x-4}\right)dx$

$= \boxed{-2\ln|x| + 5\ln|x-4| + C}$

Ex] Find $\displaystyle\int \frac{2x^2-3x-5}{\underbrace{x^3-x^2}_{x^2(x-1)}+x-1}\,dx = \int \frac{2x^2-3x-5}{(x-1)\underbrace{(x^2+1)}_{\text{irreducible quadratic}}}\,dx$

note: $x=1$ is a zero of the denom.

So $(x-1)$ must be a factor of the denom.

$x^2+1>0$, so no roots! can't factor more

How do we deal with irreducible quadratics in partial fractions?

numerator 1 deg. < than denom.

$$\frac{2x^2-3x-5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$2x^2-3x-5 = A(x^2+1) + (Bx+C)(x-1)$$

$\to x=1$: $-6 = 2A \to A=-3$

$\to x=0$: $-5 = A - C \to C=2$

$\to x=-1$: $0 = 2A + (-B+C)(-2)$

$\downarrow$

$0 = -6 + 2B - 4$

$B=5$

$$\int \left(\frac{-3}{x-1} + \frac{5x+2}{x^2+1} \right) dx$$

$$= \int \left(\underbrace{\frac{-3}{x-1}}_{-3\ln|x-1|} + \underbrace{\frac{5x}{x^2+1}}_{u=x^2+1 \; \vdots} + \underbrace{\frac{2}{x^2+1}}_{2\arctan(x)} \right) dx$$

Ex) Find $\int \frac{7x^2-13x+14}{x^3-2x^2}dx = \int \frac{7x^2-13x+14}{\underbrace{x^2}(x-2)}dx$

how do we deal w/ repeated factors?

Not like this:

$$\cancel{\frac{A}{x}+\frac{B}{x}+\frac{C}{x-2}}$$

Sorta like this:

$$\cancel{\frac{Ax+B}{x^2}+\frac{C}{x-2}}$$

Easier:

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$\frac{7x^2-13x+14}{x^2(x-2)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$7x^2-13x+14 = Ax(x-2)+B(x-2)+Cx^2$$

$\rightarrow x=2$ $\quad 16=4C \rightarrow C=4$

$\rightarrow x=0$ $\quad 14=-2B \rightarrow B=-7$

$\rightarrow x=1$ $\quad 8=-A-B+C \rightarrow A=3$

$$= \int\left(\frac{3}{x}-\frac{7}{x^2}+\frac{4}{x-2}\right)dx = \text{etc...}$$

General plan for integrating rational functions:

- If the degree of numerator $\geq$ degree of denominator: use long division.
- Then, use partial fractions.
- Edge cases:
  - If there are irreducible nonlinear factors, put a polynomial of one lower degree in numerator.

    $$\text{Ex:} \qquad \frac{}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

  - If there are repeated factors, write multiple copies, with exponents from 1 to the number of copies.

    $$\text{Ex:} \qquad \frac{}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Ex] Find $\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{u}{u^2-1}\,2u\,du = \int \frac{2u^2}{u^2-1}\,du$

$u=\sqrt{x+1} \rightarrow u^2 = x+1$

$x = u^2-1$

$du = \frac{1}{2\sqrt{x+1}}\,dx \rightarrow du = \frac{1}{2u}\,dx$

$2u\,du = dx$

$$\begin{array}{r} 2 \\ u^2-1 \,\overline{)\, 2u^2 \qquad} \\ -(2u^2-2) \\ \hline 2 \end{array}$$

$= \int \left(2 + \frac{2}{u^2-1}\right) du = 2u + \int \frac{2}{(u+1)(u-1)}\,du$

$\frac{2}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1}$

$2 = A(u-1) + B(u+1)$

$u=1:\ 2 = 2B \rightarrow B=1$

$u=-1:\ 2 = -2A \rightarrow A=-1$

$= 2u + \int \left(\frac{-1}{u+1} + \frac{1}{u-1}\right) du$

$= 2u - \ln|u+1| + \ln|u-1| + C$

$$\boxed{= 2\sqrt{x+1} - \ln\left|\sqrt{x+1}+1\right| + \ln\left|\sqrt{x+1}-1\right| + C}$$