Math 125D 2/7/24

Chapters 7.3 & 7.4

DFEP #9 Solution:

Using the shell method, we want $\int_0^4 2\pi x\sqrt{x^2+6x+25}\,dx$. Let's complete the square to get

$$2\pi\int_0^4 x\sqrt{(x+3)^2+16}\,dx$$

Setting $u = x+3$, gives $2\pi\int_3^7 (u-3)\sqrt{u^2+16}\,du$, and now we can use trigonometric substitution with $u = 4\tan(\theta)$, $du = 4\sec^2(\theta)d\theta$, leaving the integral

$$2\pi\int_{\arctan(3/4)}^{\arctan(7/4)} (4\tan(\theta)-3)\sqrt{4\sec^2\theta}4\sec^2\theta\,d\theta$$

This simplifies to:

$$16\pi\int_{\arctan(3/4)}^{\arctan(7/4)} (4\tan(\theta)\sec^3(\theta)-3\sec^3(\theta))\,d\theta$$

$$= 16\pi\left(4\sec^3(\theta)/3-\frac{3}{2}\left(\sec(\theta)\tan(\theta)+\ln|\sec(\theta)+\tan(\theta)|\right)\right)\Bigg]_{\arctan(3/4)}^{\arctan(7/4)}$$

I can't really blame you if you don't want to write all that out.

DFEP #10: Wednesday, February 7th.

Compute the area of the region bounded by the curves $y = 2x$, $y = \dfrac{2x^3+10x^2+16x+7}{x^2+5x+6}$, $x = 0$, and $x = 2$.

General plan:

To integrate...	...substitute...	...so it becomes...	...which simplifies to...
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$\sqrt{a^2 - a^2\sin^2\theta}$	$a \cdot \cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$\sqrt{a^2 + a^2\tan^2\theta}$	$a \cdot \sec\theta$ (b/c $1 + \tan^2\theta = \sec^2\theta$)
$\sqrt{x^2 - a^2}$	$x = a \cdot \sec\theta$	$\sqrt{a^2\sec^2\theta - a^2}$	$a \cdot \tan\theta$

Ex] Find $\int \frac{\sqrt{9+x^2}}{x^2}\,dx = \int \frac{\sqrt{9+9\tan^2\theta}}{9\tan^2\theta}\,3\sec^2\theta\,d\theta = \int \frac{\overbrace{\sqrt{9\sec^2\theta}}^{3\sec\theta}}{9\tan^2\theta}\,3\sec^2\theta\,d\theta = \int \frac{\cancel{9}\sec^3\theta}{\cancel{9}\tan^2\theta}\,d\theta$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{\frac{1}{\cos^3\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \int \frac{\overbrace{1}^{\sin^2\theta + \cos^2\theta}}{\cos\theta\sin^2\theta}\,d\theta = \int \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin^2\theta}\,d\theta$$

Ex] Find $\displaystyle\int \frac{\sqrt{9+x^2}}{x^2}\,dx = \int \frac{\sqrt{9+9\tan^2\theta}}{9\tan^2\theta}\,3\sec^2\theta\,d\theta = \int \frac{\sqrt{9\sec^2\theta}}{9\tan^2\theta}\,3\sec^2\theta\,d\theta = \int \frac{\cancel{9}\sec^3\theta}{\cancel{9}\tan^2\theta}\,d\theta$

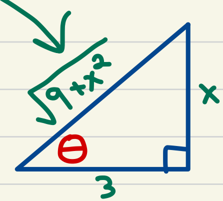
$\boxed{x = 3\tan\theta}$

$dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{\frac{1}{\cos^3\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \int \frac{\overset{\sin^2\theta+\cos^2\theta}{1}}{\cos\theta\sin^2\theta}\,d\theta = \int \frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin^2\theta}\,d\theta$$

$\tan\theta = \frac{x}{3}$

$$= \int\left(\frac{\sin^2\theta}{\cos\theta\sin^2\theta} + \frac{\cos^2\theta}{\cos\theta\sin^2\theta}\right)d\theta = \int(\sec\theta + \csc\theta\cot\theta)\,d\theta = \underbrace{\ln|\sec\theta+\tan\theta| - \csc\theta + C}_{\text{write in terms of } x}$$



$\sec\theta = \frac{\sqrt{9+x^2}}{3}$

$\tan\theta = \frac{x}{3}$

$\csc\theta = \frac{\sqrt{9+x^2}}{x}$

$$= \boxed{\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| - \frac{\sqrt{9+x^2}}{x} + C}$$

Ex] Find $\int_1^2 \frac{x}{\sqrt{x^2+4x}}\,dx = \int_1^2 \frac{x}{\sqrt{(x+2)^2-4}}\,dx = \int_{x=1}^{x=2} \frac{2\sec\theta-2}{\sqrt{4\sec^2\theta-4}}\,2\sec\theta\tan\theta\,d\theta$

complete the square

$x^2+4x+4-4$

$(x+2)^2-4$

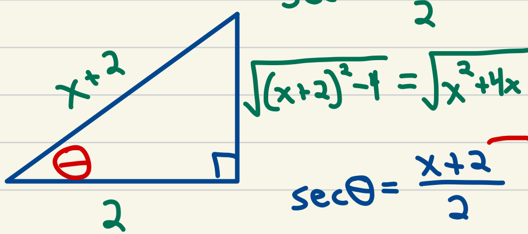
$x+2=2\sec\theta$

$dx=2\sec\theta\tan\theta\,d\theta$

$$= \int_{x=1}^{x=2} \frac{2\sec\theta-2}{\cancel{\sqrt{4\tan^2\theta}}}\,\cancel{2}\sec\theta\cancel{\tan\theta}\,d\theta$$

$$= \int_{x=1}^{x=2} \left(2\sec^2\theta-2\sec\theta\right)d\theta = 2\left(\tan\theta-\ln\left|\sec\theta+\tan\theta\right|\right)\Big]_{x=1}^{x=2}$$

$\sec\theta=\frac{x+2}{2}$



$\sec\theta=\frac{x+2}{2}$

$\tan\theta=\frac{\sqrt{x^2+4x}}{2}$

$$=2\left(\frac{\sqrt{x^2+4x}}{2}-\ln\left|\frac{x+2}{2}+\frac{\sqrt{x^2+4x}}{2}\right|\right)\Big]_1^2$$

$$\boxed{=2\left(\frac{\sqrt{12}}{2}-\ln\left|2+\frac{\sqrt{12}}{2}\right|-\frac{\sqrt{5}}{2}+\ln\left|\frac{3}{2}+\frac{\sqrt{5}}{2}\right|\right)}$$

7.4: Long Division & Partial Fractions

Polynomial

Poly. long div: a way to simplify $\frac{\text{poly}}{\text{poly}}$ when deg of num. $\geq$ deg. of denom.

Ordinary long div:

$$\begin{array}{r} 751 \\ 7\overline{)5261} \\ -49 \\ \hline 36 \\ -35 \\ \hline 11 \\ -7 \\ \hline 4 \end{array}$$

$$\frac{5261}{7} = 751 + \frac{4}{7}$$

$$\begin{array}{r} 5x^2 + 2x + 3 \\ x-3\overline{)5x^3 - 13x^2 - 3x - 3} \\ -(5x^3 - 15x^2) \\ \hline 2x^2 - 3x \\ -(2x^2 - 6x) \\ \hline 3x - 3 \\ -(3x - 9) \\ \hline 6 \end{array}$$

6 ← Remainder

Ex] Find $\displaystyle\int \frac{5x^3 - 13x^2 - 3x - 3}{x-3}\,dx$

$$= \int \Big(\overbrace{5x^2 + 2x + 3 + \frac{6}{x-3}}^{\text{much easier}} \Big)\,dx$$

$$= \frac{5}{3}x^3 + x^2 + 3x + 6\ln|x-3| + C$$

What about when deg. of num < deg. of denom? Try to factor the denominator, and then use "partial fractions" to "undo" fraction addition:

Ex) $\int \frac{3x+8}{x^2-4x}\,dx = \int \frac{3x+8}{x(x-4)}\,dx$

Want to split up $\frac{3x+8}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$

i.e. "splitting up common denominator"

next time: find constants A & B.