Math 125D 2/2/24

Chapters 7.1 & 7.2

DFEP #7 Solution:

Okay, we want the average value of $f(x) = e^{2x}(x^2 - 5x + 3)$ on the interval $[0, 2]$, so we want to find $\frac{1}{2}\int_0^2 e^{2x}(x^2 - 5x + 3)\,dx$. Let's use integration by parts! $u = x^2 - 5x + 3$, $dv = e^{2x}\,dx$, so $du = (2x - 5)\,dx$, and $v = \frac{1}{2}e^{2x}$. So:

$$\frac{1}{2}\int_0^2 e^{2x}(x^2 - 5x + 3)\,dx = \frac{1}{2}\left(\frac{1}{2}e^{2x}(x^2 - 5x + 3)\Big]_0^2 - \frac{1}{2}\int_0^2 e^{2x}(2x - 5)\,dx\right)$$

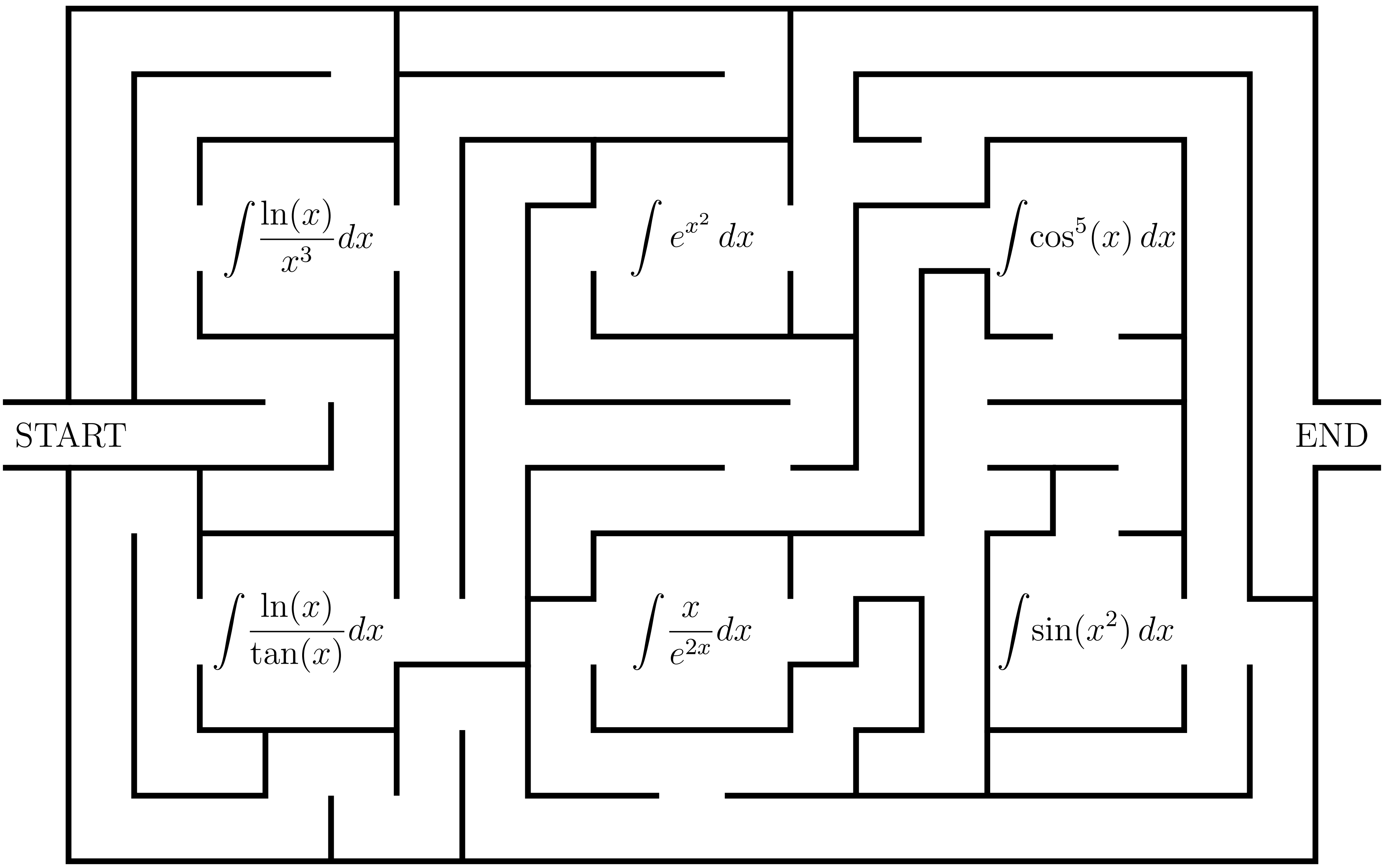
Again! $u = (2x - 5)$, $dv = e^{2x}$, etc:

$$= \frac{1}{2}\left(\left(\frac{1}{2}(e^{2x}(x^2 - 5x + 3)) - \frac{1}{4}e^{2x}(2x - 5)\right)\Bigg]_0^2 + \frac{1}{2}\int_0^2 e^{2x}\,dx\right)$$

This is fun, right? Anyway you get $-3 - e^4$.

DFEP #8: Friday, February 2nd.

Complete the following maze. But be warned: you may only pass through a room with an integral if you can evaluate it, and some of these integrals are impossible!



Ex] Find $\int_0^1 \underbrace{\sin x}_{u}\ \underbrace{e^x\,dx}_{dv}$. $= \sin x\, e^x \Big]_0^1 - \int_0^1 \cos x\, e^x\,dx$

$u = \sin x \qquad v = e^x$

$du = \cos x\,dx \qquad dv = e^x\,dx$

$= \sin(1)e - \int_0^1 \underbrace{\cos x}_{u}\ \underbrace{e^x\,dx}_{dv}$

$u = \cos x \qquad v = e^x$

$du = -\sin x\,dx \qquad dv = e^x\,dx$

$= \sin(1)e - e^x\cos x\Big]_0^1 - \underbrace{\int_0^1 \sin x\, e^x\,dx}_{\text{what we started with}} = \sin(1)e - e\cos(1) + 1 - \int_0^1 \sin x\, e^x\,dx$

$\int_0^1 \sin x\, e^x\,dx = \sin(1)e - \cos(1)e + 1 - \int_0^1 \sin x\, e^x\,dx$

$2\int_0^1 \sin x\, e^x\,dx = \sin(1)e - \cos(1)e + 1$

$\int_0^1 \sin x\, e^x\,dx = \boxed{\frac{1}{2}\left(\sin(1)e - \cos(1)e + 1\right)}$

Ex] Find $\int x^2 \underbrace{\ln(x)}_{u} dx = \frac{1}{3}\ln(x)x^3 - \int \frac{1}{3}\underbrace{x^3\left(\frac{1}{x}\right)}_{x^2}dx = \frac{1}{3}\ln(x)x^3 - \int \frac{1}{3}x^2 dx$

$u = \ln(x) \qquad v = \frac{1}{3}x^3$

$du = \frac{1}{x}dx \qquad dv = x^2 dx$

$$= \boxed{\frac{1}{3}\ln(x)\,x^3 - \frac{1}{9}x^3 + C}$$

Ex] $\int \underbrace{x}_{u}\,\underbrace{\sec^2 x\, dx}_{dv} = x\tan x - \int \tan x\, dx = x\tan x - \int \frac{\sin x}{\cos x}dx = x\tan x + \int \frac{1}{\text{☹}}d\text{☹}$

$u = x \qquad v = \tan x$

$du = dx \qquad dv = \sec^2 x\, dx$

$\text{☹} = \cos x$

$d\text{☹} = -\sin x\, dx$

$$= x\tan x + \ln|\text{☹}| + C = \boxed{x\tan x + \ln|\cos x| + C}$$

General strategy: arrange the factors of the integral from least-to-most pleasant to antidifferentiate

Good choice for u:

hard to antidiff, but easy to diff.

arcsin x

arctan x

ln x

Could go either way

x^n

Good choice for dv

Easy to antidiff.

sin x

cos x

e^x

$\sec^2 x$

sec x tan x

Chapter 7.2: Trig Integrals | Strategies for integrating products of trig functions

Four useful trig $\boxed{\text{identities}}$: True for all x : ex: $\cos^2(2x) = \frac{1}{2}(1+\cos(4x))$

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $\cos^2 x = \frac{1}{2}(1+\cos(2x))$
4. $\sin^2 x = \frac{1}{2}(1-\cos(2x))$

Half-angle identities

Come from $\cos(2x) = \cos^2 x - \sin^2 x$

General strat for finding $\int \sin^m(x)\cos^n(x)\,dx$:

(A) If m is odd, use (1) to turn all but 1 sine into cosines. Then use $u=\cos x$, $du = -\sin x\,dx$.

(B) If n is odd, use (1) to turn all but 1 cosine into sines, then let $u = \sin x$, $du = \cos x\,dx$.

(C) If m and n are both even, use (3) & (4) to reduce the powers.

Ex] Compute $\int \sin^3 x\,dx$

$$= \int \sin x \sin^2 x\,dx \overset{(1)}{=} \int \sin x(1-\cos^2 x)\,dx$$

$u = \cos x$
$du = -\sin x\,dx$

$$= -\int (1-u^2)\,du = -u + \frac{1}{3}u^3 + C = \boxed{-\cos x + \frac{1}{3}\cos^3 x + C}$$

Ex] Find $\int \sin^4 x \cos^5 x \, dx$

① $\sin^2 x + \cos^2 x = 1$

② $\tan^2 x + 1 = \sec^2 x$

③ $\cos^2 x = \frac{1}{2}(1+\cos(2x))$

④ $\sin^2 x = \frac{1}{2}(1-\cos(2x))$

Case Ⓑ: turn all but 1 cosine into sines.

$$= \int \sin^4 x \cos^4 x \cos x \, dx = \int \sin^4 x (\cos^2 x)^2 \cos x \, dx$$

$$\overset{①}{=} \int \sin^4 x (1-\sin^2 x)^2 \cos x \, dx = \int u^4 (1-u^2)^2 \, du = \int u^4 (1-2u^2+u^4) \, du$$

$u = \sin x$
$du = \cos x \, dx$

$$= \int (u^4 - 2u^6 + u^8) \, du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$

$$= \boxed{\frac{1}{5}\sin^5(x) - \frac{2}{7}\sin^7(x) + \frac{1}{9}\sin^9 x + C}$$

Ex) $\int \sin^2 x \cos^2 x \, dx$

3 & 4

$$= \int \frac{1}{2}(1-\cos(2x))\,\frac{1}{2}(1+\cos(2x))\,dx$$

$$= \frac{1}{4}\int (1-\cos^2(2x))\,dx \overset{1}{=} \frac{1}{4}\int \underbrace{\sin^2(2x)}_{\text{even powers again!}}\,dx$$

$$\overset{4}{=} \frac{1}{4}\int \frac{1}{2}(1-\cos(4x))\,dx$$

$$= \frac{1}{8}\int (1-\cos(4x))\,dx = \boxed{\frac{1}{8}\left(x - \frac{1}{4}\sin(4x)\right) + C}$$

$$\int \cos(4x)\,dx = \frac{1}{4}\int \cos(u)\,du = \frac{1}{4}\sin(4x)+C$$

$u = 4x$, $du = 4x$

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $\cos^2 x = \frac{1}{2}(1+\cos(2x))$
4. $\sin^2 x = \frac{1}{2}(1-\cos(2x))$

What about integrals like $\int \tan^m x \sec^n x \, dx$?

General idea: if we want $u = \tan x$, need $du = \sec^2 x \, dx$, so 2 powers $\sec x$ left over.

(To be continued)