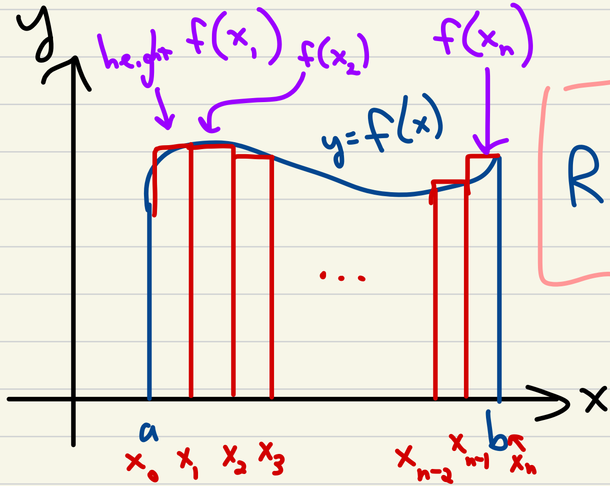
Math 125D 1/8/24

Chapters 5.1 & 5.2

From last time:

General way to find these Riemann sums in Σ-notation.



widths: $\frac{b-a}{n} = \Delta x$

$x_0 = a$

$x_1 = a + \Delta x = a + \frac{b-a}{n}$

$x_2 = a + 2\Delta x = a + 2\left(\frac{b-a}{n}\right)$

$$R_n = \sum_{i=1}^{n} \underbrace{\Delta x}_{\text{width}} \underbrace{f(x_i)}_{\text{height}}$$

where $\Delta x = \frac{b-a}{n}$

and $x_i = a + i\Delta x$

$$L_n = \sum_{i=0}^{n-1} \Delta x f(x_i)$$

If f is a "nice" function,

then $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$,

so either limit gives the area.

Or, more generally you can use any point in the i^{th} subinterval to choose the heights. These are called "sample points" x_i^*

$$\lim_{n\to\infty} \sum_{i=0}^{\infty} \Delta x f(x_i^*)$$

any sample points.

Chapter 5.2: The Definite Integral

$\Delta x = \frac{b-a}{n}$ — sample points btwn $a+(i-1)\Delta x$ and $a+i\Delta x$

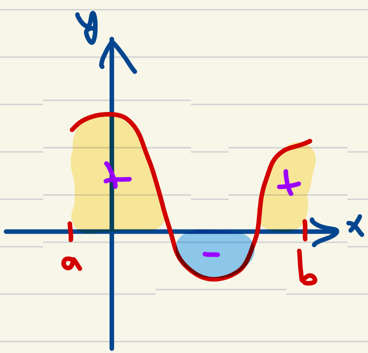
Definition The definite integral $\int_a^b f(x)\,dx$ is defined as $\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i^*)$,

assuming that the limit is defined <u>and</u> that it's the same for all choices of x_i^*s.

Or, more intuitively: $\int_a^b f(x)\,dx$ is the area* under $y=f(x)$ and above the x-axis btwn $x=a$ & $x=b$.

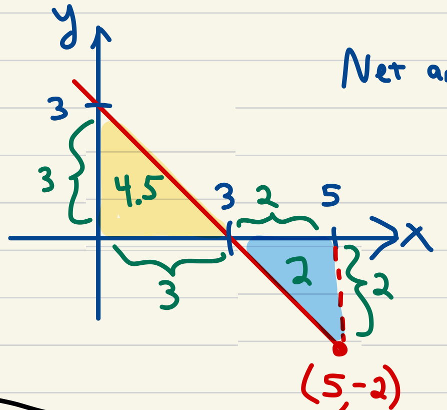
* "Net area":

areas below x-axis count as "negative area"



$$\underbrace{\int}_{\text{Summa}}\overbrace{{}_a^b}^{\text{bounds of integration}}\underbrace{f(x)}_{\text{integrand}}\,\underbrace{dx}_{\text{very tiny change in } x}$$

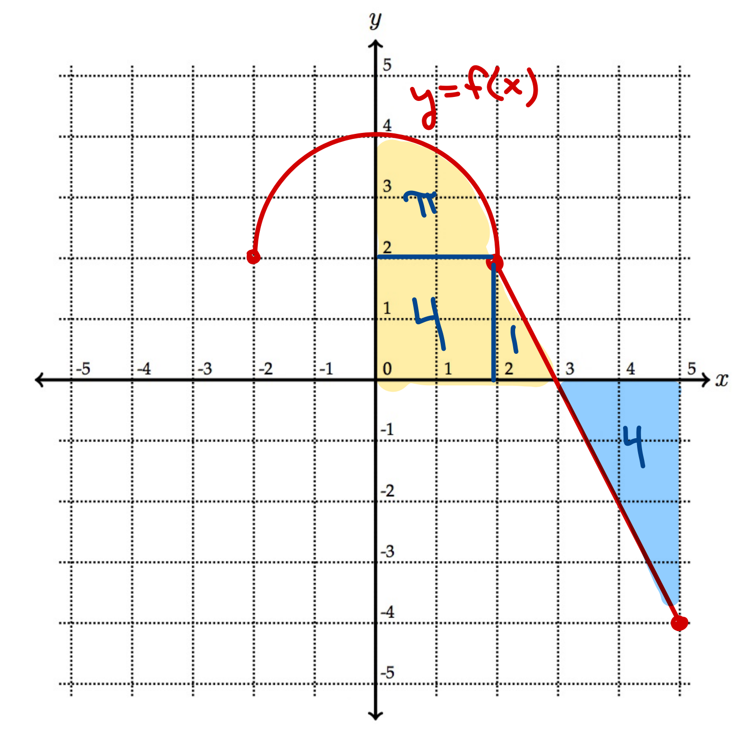
Ex] Find $\int_0^5 (3-x)\,dx$.



Net area $= 4.5 - 2 = \boxed{2.5}$

Ex] Let $f(x) = \begin{cases} 2+\sqrt{4-x^2} & \text{if } -2 \le x \le 2 \\ 6-2x & \text{if } 2 < x \le 5 \end{cases}$

upper-semicircle
center $(0,2)$
radius 2



Find $\int_0^5 f(x)\,dx$

Net area $= \pi + 4 + 1 - 4 = \boxed{\pi + 1}$

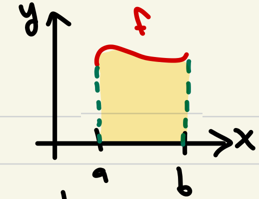
$y = 2 + \sqrt{4-x^2}$

$y - 2 = \sqrt{4-x^2}$

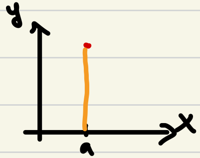
$x^2 + (y-2)^2 = 4$

Properties of Definite Integrals

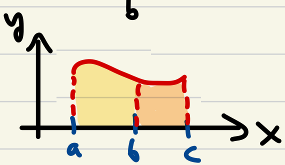
Reminder: $\int_a^b f(x)\,dx$ is this area:



① $\int_a^a f(x)\,dx = 0$



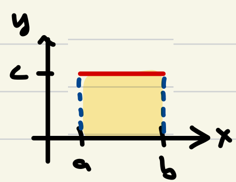
② $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



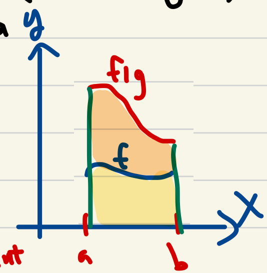
③ $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

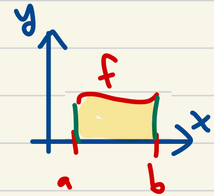
④ $\int_a^b c\,dx = c(b-a)$

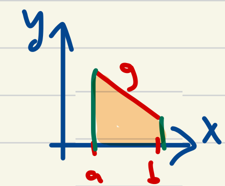
constant



⑤ $\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$







constant

⑥ $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

⑦ If $f(x) \geq g(x)$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

⑧ If $m \leq f(x) \leq M$ then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Constants