Math 125D 1/5/24

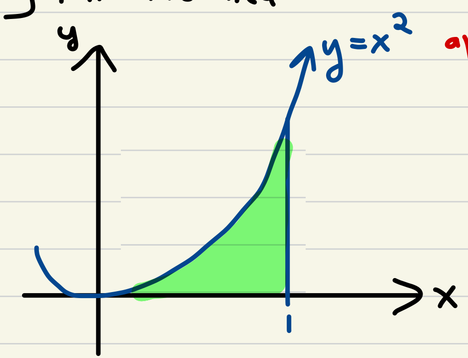
Chapter 5.1

After class today: Field trip to the Math Study Center!

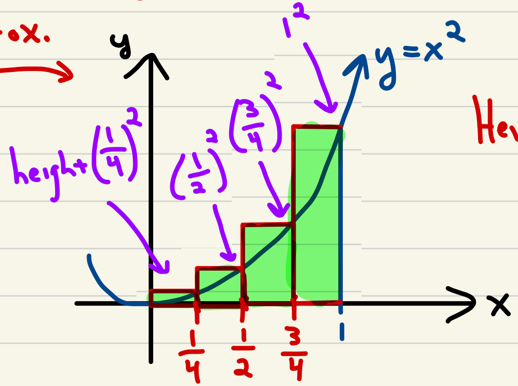
Chapter 5.1: Area Problems and Riemann Sums

Question: How can we find the area between $y=f(x)$ and the x-axis over an interval $[a,b]$?

Ex] Find this area:



approx.



Cut interval into 4 pieces, and use upper-right corners to draw rectangles.

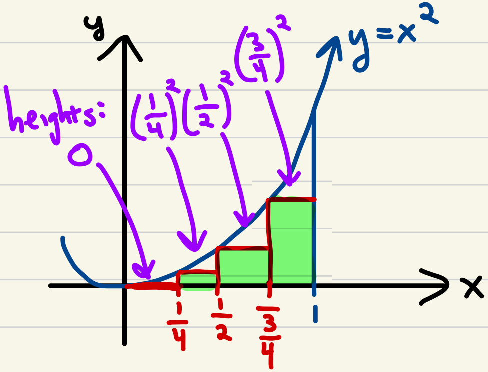
Height of each rectangle: $f(\text{rightmost coord.})$

$$\text{Total area} = R_4 = \underbrace{\left(\tfrac{1}{4}\right)}_{\text{base}}\underbrace{\left(\tfrac{1}{4}\right)^2}_{\text{height}} + \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^2 + \left(\tfrac{1}{4}\right)(1)^2 = \underbrace{0.46875}_{\text{overestimate}}$$

Right-hand Riemann sum w/ 4 subintervals

(rectangles poke out over graph)

What if we use left-hand side?



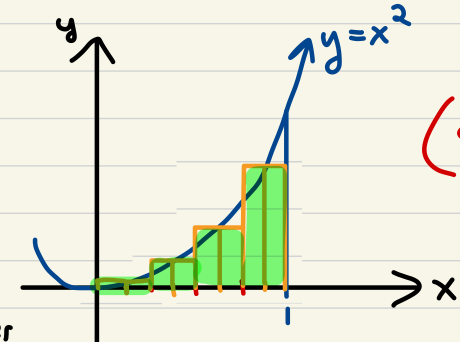
left-hand Riemann sum

$$L_4 = \left(\tfrac{1}{4}\right)(0) + \left(\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^2 = 0.21875$$

underestimate

You could also compute M_4:

use the midpoint of each subinterval to find heights.



(Something to think about: when is M_n an over/under-estimate?

Idea: To get an **exact** answer,

let # of rectangles $\to \infty$.

Σ-notation: A better way to write big sums.

means: add 2^i for each integer i from 0 to 10

Ex] "$1+2+4+8+\dots+1024$" is hard to work with. Instead, write $\boxed{\sum_{i=0}^{10} 2^i}$ $= 2^0+2^1+\dots+2^{10}$

$$\sum_{i=0}^{10} 2^i$$

find value: 10

what terms are being added: 2^i

the "indexing variable": i

starting value: 0

Ex] $\sum_{k=3}^{7} k^2$ means ...

$$\boxed{3^2+4^2+5^2+6^2+7^2}$$

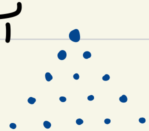
Ex] $2^2+4^2+6^2+8^2 = \sum_{k=1}^{4}(2k)^2$

Four summation formulas:

$$\sum_{i=1}^{n} 1 = n$$

means $\underbrace{1+1+1+\dots+1}_{n \text{ times}}$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$



$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Two Summation Properties:

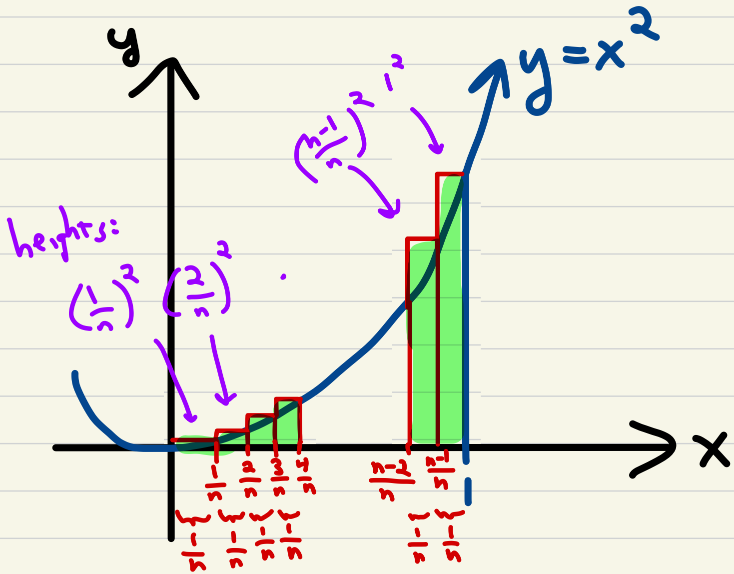
$$\sum_{i=1}^{n} c\,\underbrace{a_i}_{\text{whatever}} = c\sum_{i=1}^{n} a_i$$

constant: c

(like: $ca+cb = c(a+b)$)

$$\sum_{i=1}^{n}(a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

What if we use $\underline{\underline{n}}$ subintervals?



$$R_n = \overbrace{\left(\frac{1}{n}\right)}^{\text{base}}\overbrace{\left(\frac{1}{n}\right)^2}^{\text{height}} + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \dots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$$

$$\sum_{i=1}^{n}\left(\frac{1}{n}\right)\left(\frac{i}{n}\right)^2 \xrightarrow{\text{simplify}} = \sum_{i=1}^{n}\frac{i^2}{n^3}$$

constant (doesn't depend on variable of the summation)

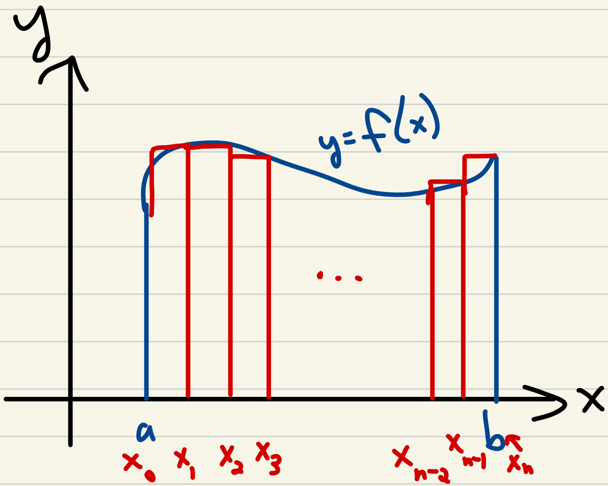
$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{n(n+1)(2n+1)}{6n^3}$$

As $n \to \infty$ this is $\displaystyle\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty}\frac{2n^3 + \text{don't care}}{6n^3} = \frac{2}{6} = \boxed{\frac{1}{3}}$

Math 124 fact: $\displaystyle\lim_{n\to\infty}\frac{\text{polynom.}}{\text{polynom.}} = \lim_{n\to\infty}\frac{\text{highest deg. term}}{\text{highest deg. term}}$

General way to find these Riemann sums in Σ-notation.



$$\sum_{i=1}^{n} \Delta x \underline{\qquad\qquad}$$

To be continued

widths: $\frac{b-a}{n} = \Delta x$

$x_0 = a$

$x_1 = a + \Delta x = a + \frac{b-a}{n}$