Math 125D 1/31/24

Chapters 6.5 & 7.1

DFEP #6 Solution:

Let's imagine cutting the soup into horizontal cross sections (it's a very thick soup, I guess): how much work does it take to bring each cross section to the top?

Say $y = 0$ is the bottom of the pot and $y = 0.25$ is the top of the pot. (We're working in meters.) At a depth of y meters from the top, we get a slice with area $\pi\left(\sqrt{.25^2 - y^2}\right)^2 = \pi(.25^2 - y^2)$ (hint: to see this, use the Pythagorean theorem). So the mass at that height is $\pi(.25^2 - y^2)(1500)\,\Delta y$, and the work required is $9.8 \cdot 1500\pi(.25^2 - y^2)y\,\Delta y$.

So the total work required is

$$\begin{aligned}
&\int_0^{0.25} 9.8 \cdot 1500\pi(.25^2 - y^2)y\,dy \\
&= 9.8 \cdot 1500\pi \int_0^{0.25} \left(-y^3 + \frac{y}{16}\right) dy \\
&= 9.8 \cdot 1500\pi \left(\frac{-y^4}{4} + \frac{y^2}{32}\right)\Bigg]_0^{0.25} \\
&= 9.8 \cdot 1500\pi \left(\frac{-1}{1024} + \frac{1}{512}\right) \\
&= \frac{9.8 \cdot 1500\pi}{1024}\ \mathrm{J}
\end{aligned}$$

DFEP #7: Wednesday, Janury 31st.

Determine the average value of the function $f(x) = e^{2x}(x^2 - 5x + 3)$ on the interval $[0, 2]$.

Chapter 6.5: Average Value of a Function

$\Delta x = \frac{b-a}{n}$ $\quad n\Delta x = b-a$

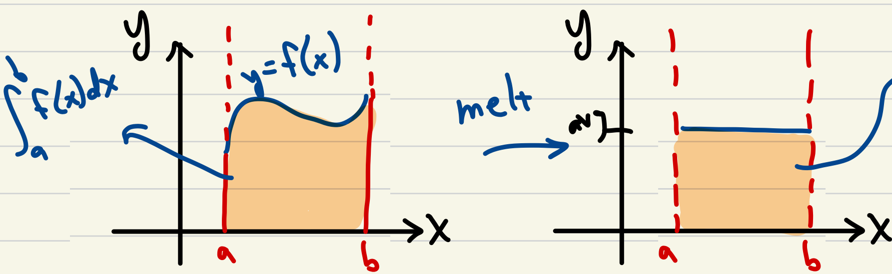
How do we find the average of f on the interval $[a,b]$?

Idea: Split $[a,b]$ into n pieces, then average values of random pts in those pieces.

$$\text{Average} \approx \frac{f(x_1^*) + f(x_2^*) + \cdots f(x_n^*)}{n} = \sum_{i=1}^{n} \frac{1}{n} f(x_i^*)$$

$$\text{Average} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n\Delta x} f(x_i^*)\Delta x = \lim_{n\to\infty} \left(\frac{1}{b-a}\right) \sum_{i=1}^{n} f(x_i^*)\Delta x = \boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

Or: Imagine you draw $\int_a^b f(x)\,dx$ as an area, then melt it.



area $= (b-a)(\text{avg.})$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{\cancel{(b-a)}(\text{avg})}{\cancel{b-a}}$$

Ex] What is the average value of $\sin x$ on $[0, \frac{\pi}{2}]$?

$$\frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2} \sin x\, dx = \frac{2}{\pi}\left(-\cos x\right]_0^{\pi/2} = \frac{2}{\pi}\Big(\underbrace{-\cos\left(\tfrac{\pi}{2}\right)}_{0} + \underbrace{\cos(0)}_{1}\Big) = \boxed{\frac{2}{\pi}}$$

Next few chapters: Lots of new integration techniques.

Chapter 7.1: Integration by Parts

Deriv rule		Integration rule
Chain rule	$\longrightarrow$	u-substitution
Product rule		IBP

Say $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$.

Product rule: $\frac{d}{dx}\left(F(x)G(x)\right) = f(x)G(x) + F(x)g(x)$

Equivalently: $F(x)G(x) = \int f(x)G(x)\,dx + \int F(x)g(x)\,dx$

New names:
- $F(x) = u$
- $f(x)\,dx = du$
- $G(x) = v$
- $g(x)\,dx = dv$

$$uv = \int v\,du + \underbrace{\int u\,dv}_{\text{solve for this}} \longrightarrow \boxed{\int u\,dv = uv - \int v\,du}$$

$\longleftarrow$ Integration by parts

(hopefully this is easier than $\int u\,dv$)

$$\text{IBP: } \int u\,dv = uv - \int v\,du.$$

Process:

1. Split your integral into "u" and "dv" parts.
2. Find du (by differentiating u) and v (by integrating dv).
3. Apply IBP. Solve the result.

Hopefully, u becomes simpler when diff'ed and/or dv becomes simpler when integrated.

Ex] Compute $$\int \underbrace{x}_{u}\,\underbrace{\cos(x)\,dx}_{dv} = \overbrace{x\sin x}^{uv} - \overbrace{\int \sin x\,dx}^{\int v\,du} = \boxed{x\sin x + \cos x + C}$$

$$u = x \qquad v = \sin x$$

$$du = dx \qquad dv = \cos x\,dx$$

(Very common to let powers of x be u)

Ex] Find $\int \underbrace{\arcsin x}_{u}\,\underbrace{dx}_{dv}$. $= x\arcsin x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx = x\arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{☺}}\,d☺$

$u = \arcsin x \qquad v = x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad dv = dx$

$☺ = 1 - x^2$

$d☺ = -2x\,dx$

$= x\arcsin x + \frac{2}{2}\sqrt{☺} + C$

$= \boxed{x\arcsin x + \sqrt{1-x^2} + C}$

Note: This same idea also works for $\int \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv}$ and $\int \underbrace{\arctan x}_{u}\,\underbrace{dx}_{dv}$

Ex] Find $\int_1^2 \underbrace{x}_{u}\,\underbrace{e^x dx}_{dv} = x e^x\Big]_1^2 - \int_1^2 e^x\,dx = 2e^2 - e - \left(e^x\right)\Big]_1^2 = \cancel{2}e^2 \cancel{- e} - \cancel{e^2} \cancel{+ e}$

$= \boxed{e^2}$

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x\,dx$

When using IBP on a def int, evaluate uv at bounds

Ex] Find $\int_0^1 \underbrace{\sin x}_{u}\, \underbrace{e^x dx}_{dv}$. $= \sin x\, e^x \Big]_0^1 - \int_0^1 \cos x\, e^x\, dx$

$u = \sin x \quad v = e^x$

$du = \cos x\, dx \quad dv = e^x\, dx$

$= \sin(1)e - \int_0^1 \underbrace{\cos x}_{u}\, \underbrace{e^x dx}_{dv}$

$u = \cos x \quad v = e^x$

$du = -\sin x\, dx \quad dv = e^x\, dx$

$= \sin(1)e - e^x \cos x \Big]_0^1 - \underbrace{\int_0^1 \sin x\, e^x\, dx}_{\text{what we started with}}$

(To be continued)