Math 125D 1/29/24

Chapter 6.4

DFEP #5 Solution:

On the interval $[1, 4]$, we know that $\dfrac{x-1}{x} < 1 < e \le e^x$, so $y = \dfrac{x-1}{x}$ is always below $y = e^x$. That means the volume is given by:

$$\int_1^4 \pi \left((e^x + 1)^2 - \left(\frac{x-1}{x} + 1 \right)^2 \right) dx$$

This isn't too bad. (To integrate e^{2x}, just use $u = 2x$.)

$$\begin{aligned}
&\pi \int_1^4 \left(e^{2x} + 2e^x + 1 - \frac{4x^2 - 4x + 1}{x^2} \right) dx \\
&= \pi \int_1^4 \left(e^{2x} + 2e^x + 1 - 4 - \frac{4}{x} + \frac{1}{x^2} \right) dx \\
&= \pi \left(\frac{1}{2} e^{2x} + 2e^x - 3x - 4 \ln|x| - \frac{1}{x} \right) \Big]_1^4 \\
&= \pi \left(\frac{1}{2} e^8 + 2e^4 - 12 - 4 \ln|4| - \frac{1}{4} \right) - \pi \left(\frac{1}{2} e^2 + 2e - 3 - 0 - \frac{1}{1} \right)
\end{aligned}$$

DFEP #6: Monday, January 29th.

A hemispherical pot with diameter 50 cm is filled to the brim with tomato soup of uniform density 1500 kg/m^3. Find the work required to drink all of the soup with a straw. (The top of the straw is level with the rim of the tank.)

6.4: Work

$\underset{N}{\text{Force}} = \underset{kg}{\text{mass}} \cdot \underset{m/s^2}{\text{acceleration}}$

On Earth's surface: acceleration due to gravity $\approx 9.8\ m/s^2$

$\underset{\substack{J \\ \text{Joule}}}{\text{Work}} = \underset{N}{\text{Force}} \cdot \underset{m}{[\text{distance traveled in direction of force}]}$

Ex) Find the work needed to lift an 8-kg cat 1.5 m up.

$$\text{Force} = \underbrace{(8\ kg)}_{\text{mass}}\underbrace{(9.8\ m/s^2)}_{\text{acceleration due to gravity}}$$

$$\text{Work}: \underbrace{(8 \cdot 9.8\ N)}_{\text{force}}\underbrace{(1.5\ m)}_{\text{dist}} = \boxed{(12 \cdot 9.8)\ J}$$

Why would you need calculus for this?

1. Different parts of the object are being moved by different amounts.
2. The force applied is changing over the course of moving the object.

Solution: look at small slices, find work done on each, take limit as # slices $\to \infty$. Get an integral.

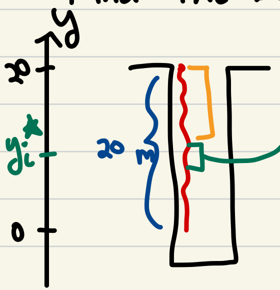
Ex] A 20-m rope is dangling from the top of a pit. The rope has a linear density of $2\ \text{kg/m}$.

Find the work needed to pull it out of the pit.

$$\underbrace{2\ \text{kg/m}}_{\frac{\text{mass}}{\text{length}}}$$

Whole rope: $(20 \cdot 2) = 40\ \text{kg}$

3 m of rope: $(3 \cdot 2) = 6\ \text{kg}$



slice w/ thickness Δy.

Work needed for this slice:

$$W = \overbrace{(\underbrace{2\Delta y}_{m} \cdot \underbrace{9.8}_{a})}^{\text{force}}\overbrace{(20 - y_i^*)}^{\text{dist}}$$

$$\underbrace{(\text{length of slice})}_{\Delta y}\underbrace{(\text{density})}_{2}$$

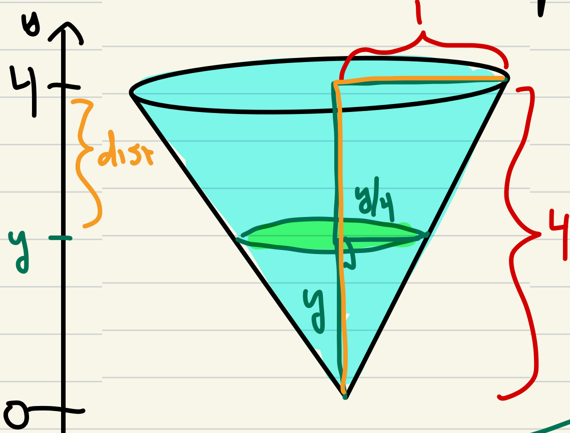
$$\text{Total} \approx \sum_{i=1}^{n} (2\,\Delta y \cdot 9.8)(20 - y_i^*)$$

$$\Big\downarrow \lim_{n \to \infty}$$

$$\int_0^{20} \big(2 \cdot 9.8 \cdot (20 - y)\big)\,dy = 2 \cdot 9.8\left(20y - \frac{1}{2}y^2\right)\Big]_0^{20} = \boxed{2 \cdot 9.8 \cdot (200)\ \text{J}}$$

Ex] An inverted cone w/ height 4 m & radius 1 m is filled w/ water w/ density 1000 kg/m^3.

Find the work needed to pump all water to of the cone.



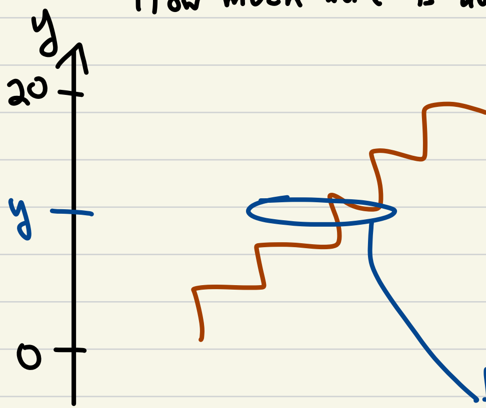
$$\int_0^4 \overbrace{\Big(\overbrace{\underbrace{\left(\left(\pi\left(\tfrac{y}{4}\right)^2\right)(1000)\right)}_{\text{mass}}\underbrace{(9.8)}_{\text{acc}}}^{\text{force}}\underbrace{(4-y)}_{\text{dist}}\Big)}^{\text{work to move tiny slice}}dy$$

$$\text{mass} = \text{volume} \cdot \overbrace{\text{density}}^{1000\ kg/m^3}$$

$$\text{volume} = \underbrace{(\text{area of circle})}_{\pi\left(\frac{y}{4}\right)^2}\overbrace{(\text{height})}^{dy}$$

Ex) A leaky 5-lb. bucket is carried up a 20-ft flight of stairs. At first, there's 10 lbs of water in the bucket. It leaks at a constant rate and there's only 2 lbs left at the top.

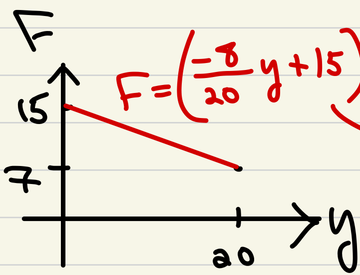
How much work is done?



lbs is already a unit of force.

$$\underbrace{\text{Work}}_{\text{ft-lbs}} = \underbrace{\text{Force}}_{\text{lbs}} \cdot \underbrace{\text{dist}}_{\text{ft}}$$

How much work do we need to move it past here?



$$F = \left(\frac{-8}{20}y + 15\right)$$

$$\text{Work} = \text{force} \cdot \overbrace{\text{dist}}^{dy}$$

$$\int_0^{20} \left(\frac{-2}{5}y + 15\right) dy$$