Math 125D 1/22/24

Chapters 6.1 & 6.2

DFEP #3 Solution:

(a) Set $u = \ln(x^2)$, $du = \dfrac{2x}{x^2}\,dx = \dfrac{2}{x}\,dx$.

So $\displaystyle\int \frac{1}{x\ln(x^2)}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|\ln(x^2)| + C.$

(b) Rewrite as $\displaystyle\int e^x e^{e^x}\,dx$. Then $u = e^x$, $du = e^x\,dx$ gives:

$\displaystyle\int e^u\,du = e^u + C = e^{e^x} + C.$

(c) Let $u = x^2 - 16$, so $x^4 = (u+16)^2$ so the integral becomes

$$\frac{1}{2}\int_{-16}^{-7} (u+16)^2\sqrt[3]{u}\,du$$

Expand to get

$$\frac{1}{2}\int_{-16}^{-7} \left(u^{7/3} + 32u^{4/3} + 256u^{1/3}\right)\,du$$

which is solved easily enough by the power rule:

$$\frac{1}{2}\left[\frac{3u^{10/3}}{10} + \frac{96u^{7/3}}{7} + 192u^{4/3}\right]_{-16}^{-7} \approx -254.1$$

DFEP #4: Monday, January 22nd.

Find a such that the area of the region in the first quadrant bounded by $y = x^3$ and $y = ax$ is 5.

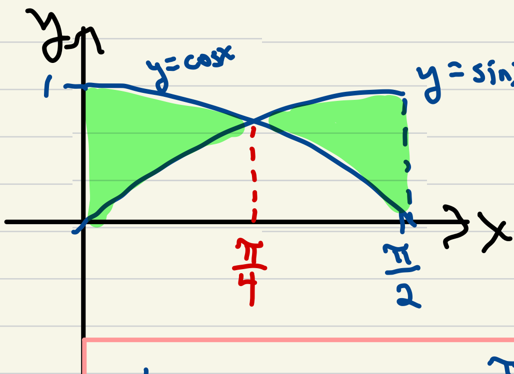
A List of Topics for the First Midterm

Here's what you should be able to do for the midterm this week.

1. Riemann sums
   (a) Compute L_n, R_n, and M_n estimates for areas under curves.
   (b) Write the (exact) area under a curve as a limit of Riemann sums and (for certain curves) evaluate that limit.
   (c) Recognize such a limit, convert it to an integral, and compute it.
2. Integration
   (a) Find antiderivatives of certain elementary functions including polynomials, exponential functions, and certain trigonometric functions.
   (b) Use u-substitution to evaluate more challenging integrals.
   (c) Compute indefinite integrals and definite integrals.
   (d) Evaluate integrals of odd or even functions on intervals of the form $[-a, a]$.
   (e) Use the fundamental theorem of calculus to differentiate functions that are defined in terms of integrals.
3. Applications
   (a) Given velocity or acceleration, compute the net displacement of an object over a time interval *or* compute its total distance traveled.
   (b) Find the area bounded by two or more curves in the plane.

6.1, Continued

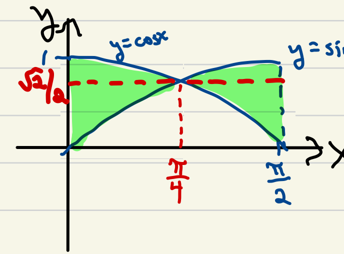
Ex] Find the area btwn $y=\sin x$ and $y=\cos x$ on the domain $[0, \frac{\pi}{2}]$.



$\sin x = \cos x$

$\tan x = 1$

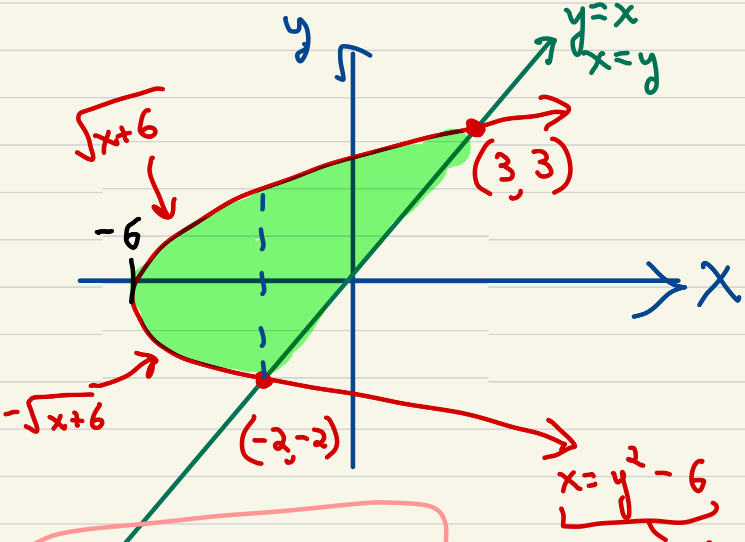
$x = \frac{\pi}{4}$



$$\boxed{\int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx}$$

or: $$\boxed{2\int_0^{\pi/4} (\cos x - \sin x)\,dx}$$

Ex] Find area btwn $x=y^2-6$ and $y=x$.



$$x = x^2 - 6$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3 \text{ or } -2$$

$$\underbrace{x = y^2 - 6}_{x+6=y^2}$$

$$y = \pm\sqrt{x+6}$$

$$\int_{-2}^{3} \Big(\underset{\text{right}}{y} - \underset{\text{left}}{(y^2-6)} \Big)\, dy \longrightarrow \text{compute}$$

or

$$\int_{-6}^{-2} \left(\sqrt{x+6} - \left(-\sqrt{x+6}\right) \right) dx + \int_{-2}^{3} \left(\sqrt{x+6} - x \right) dx$$

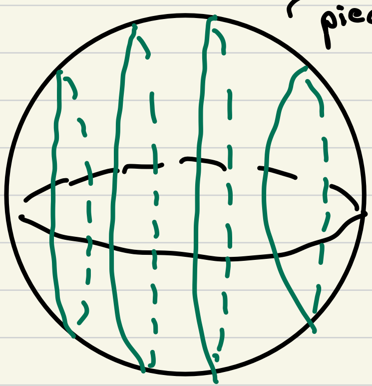
Chapter 6.2: Volume & Solids of Revolution

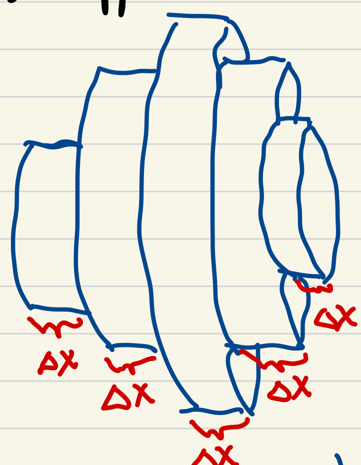
To compute a volume of a solid from $x=a$ to $x=b$, calculate $\displaystyle\int_a^b \underbrace{A(x)}_{\text{cross-sectional area}}\,dx$

How can we use integrals to compute volume?

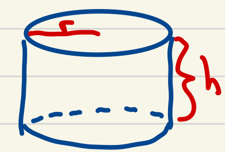
Ex] How can we find the volume of a sphere?

Idea: $\xrightarrow[\text{pieces}]{\text{cut into 5}}$ approximate as thin cylinders:





Volume of cylinder:



Volume $=(\text{Area})h = \pi r^2 h$

$$\text{Volume} \approx \sum_{i=1}^{n} A(x_i)\,\Delta x \xrightarrow{\text{take limit}} \text{Volume} = \int_a^b \overbrace{A(x)}^{\text{area of slice at } x}\,dx$$

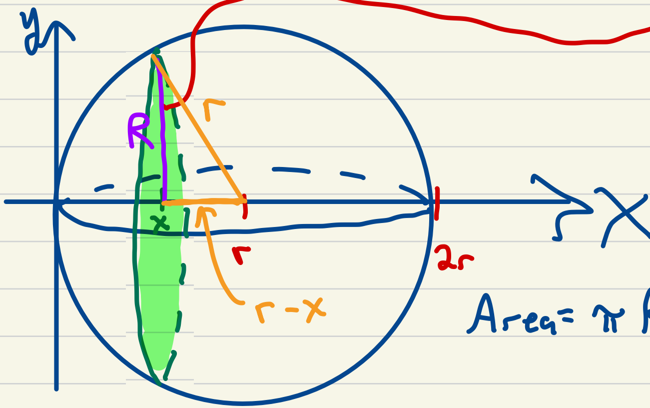
area of each slice

largest & smallest x-coords

Ex] Find volume of a sphere w/ radius r.

area of slice

$$V = \int_0^{2r} \qquad dx = \int_0^{2r} \pi\left(2rx - x^2\right) dx$$



$$\text{Area} = \pi R^2 = \pi\left(2rx - x^2\right)$$

$$= \pi\left(rx^2 - \frac{1}{3}x^3\right)\Big]_0^{2r}$$

$$R^2 + (r-x)^2 = r^2$$

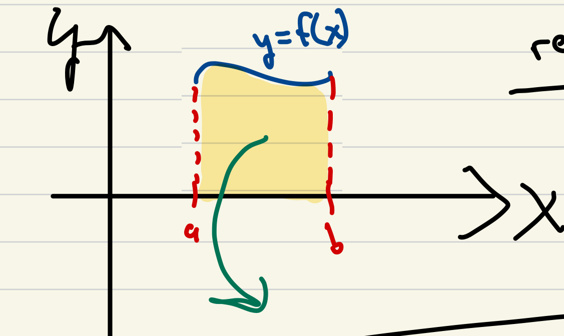
$$R^2 = r^2 - (r-x)^2 = r^2 - \left(r^2 - 2rx + x^2\right) = 2rx - x^2$$

$$= \pi\left(4r^3 - \frac{8}{3}r^3\right)$$

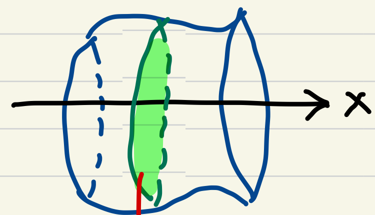
$$= \boxed{\frac{4}{3}\pi r^3}$$

In general, solids of revolution:

Say you take the region under $y=f(x)$ and above $y=0$ from $x=a$ to $x=b$, and revolve it around x-axis:



result



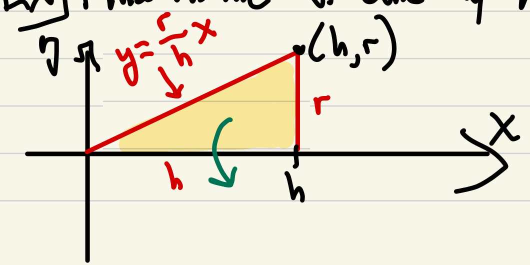
Area $=\pi(f(x))^2$

("Disc method")

The resulting volume is:

$$\int_a^b \pi (f(x))^2 \, dx$$

Ex) Find volume of cone w/ height h & radius r.



$$\int_0^h \pi\left(\frac{r}{h}x\right)^2 dx = \frac{\pi r^2}{h^2}\left(\frac{1}{3}x^3\right)\Big]_0^h = \frac{\pi r^2 h^{\cancel{3}}}{3\cancel{h^2}} = \boxed{\frac{1}{3}\pi r^2 h}$$