Math 125D 1/17/24

Chapter 5.5

DFEP #1 Solution:

(a) We want to write $\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{i\pi}{3n}\right)\frac{\pi}{3n}$ as an integral. Recall that using a limit of right-hand Riemann sums, we have $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$, where $\Delta x = \frac{b-a}{n}$.

Stare at this limit a bit and you can see it's what you get when $a=0$, $b=\frac{\pi}{3}$, and $f(x)=\cos(x)$.

So it's

$$\int_0^{\pi/3}\cos(x)\,dx = \sin(x)\Big]_0^{\pi/3} = \sin\left(\frac{\pi}{3}\right)-\sin(0)=\frac{\sqrt{3}}{2}.$$

(b) Hey, I warned you, this one is tough. First, a little algebra:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{n^2-\frac{i^2}{4}}} = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n\sqrt{1-\left(\frac{i}{2n}\right)^2}}$$

That $\frac{i}{2n}$ sure looks like it wants to be x_i, which means Δx should be $\frac{1}{2n}$. So rewrite it as:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{2}{\sqrt{1-\left(\frac{i}{2n}\right)^2}}\cdot\frac{1}{2n}$$

This is the limit of left-hand Riemann sums for the integral

$$\int_0^{\frac{1}{2}}\frac{2}{\sqrt{1-x^2}}\,dx = 2\arcsin(1/2)-2\arcsin(0)=\frac{\pi}{3}$$

DFEP #2: Wednesday, January 17th.

Consider the function $f(x)=\int_{\sin(2x)}^{3x} e^{t^2}\,dt$.

Is $f(x)$ concave up or concave down at $x=\frac{\pi}{2}$?

Chapter 5.5: u-substitution

Chain rule, backwards.

Ex] What is $\int 2x e^{x^2} dx$? It's $\boxed{e^{x^2} + C}$. Why? $\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2} \cdot (2x)$

chain rule

Note: $\frac{d}{dx}(x^2) = 2x$

How can we notice situations where we can "undo" the chain rule?

Idea: $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\, g'(x)$. Look for (a) A function $g(x)$ that's "plugged into" another,
(b) A factor of $g'(x)$.

Process: Take that "inner function" $g(x)$, and call it u.

If $g(x) = u$, then $g'(x) = \frac{du}{dx} \rightarrow du = g'(x)\,dx$

Replace all 'x's in the integral with 'u's, using these two equations.

Convert final answer back into terms of x.

Ex] $\int 2x e^{x^2} dx = \underbrace{\int e^u du}_{\text{much easier}}$

$u = x^2$

diff

$du = 2x\,dx$

$= e^u + C = \boxed{e^{x^2} + C}$

Ex] Find $\int \sqrt{\sin x}\,\cos x\,dx$. $= \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{2u^{3/2}}{3} + C = \boxed{\frac{2(\sin x)^{3/2}}{3} + C}$

$u = \sin x$

$du = \cos x\,dx$

Ex] Find $\int 4\sqrt{4x+1}\,dx = \int \sqrt{u}\,du = \frac{2u^{3/2}}{3} + C = \frac{2(4x+1)^{3/2}}{3} + C$

$u = 4x+1$

$du = 4dx$

Ex] Find $\int \cos(\pi x)\,dx = \int \frac{1}{\pi}\cos(u)\,du = \frac{1}{\pi}\sin(u) + C = \boxed{\frac{1}{\pi}\sin(\pi x) + C}$

$u = \pi x$

$du = \pi\,dx$

$\frac{du}{\pi} = dx$

or: $\frac{1}{\pi}\int \cos(\pi x)\,\pi\,dx = \frac{1}{\pi}\int \cos(u)\,du = \frac{1}{\pi}\sin(u) + C \ldots$

$u = \pi x$

$du = \pi\,dx$

(this only worked because the "missing factor" was a constant)

Ex) Find $\int \frac{1+x}{1+x^2}\,dx \xrightarrow{\text{split}} = \int \underbrace{\frac{1}{1+x^2}dx}_{\text{easy}} + \frac{1}{2}\int \frac{2x}{1+x^2}dx = \arctan(x) + \frac{1}{2}\int \frac{1}{u}\,du$

$= \arctan(x) + \frac{1}{2}\ln|u| + C$

$\boxed{= \arctan(x) + \frac{1}{2}\ln|1+x^2| + C}$

$u = 1+x^2$

$du = 2x\,dx$

the "1+" part keeps us from factoring out $2x\,dx$

(u-sub doesn't work yet)

$u = 1+x^2$

$du = 2x\,dx$

Ex) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = \int \frac{-1}{u}du = -\ln|u| + C = \boxed{-\ln|\cos x| + C}$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

Ex) $\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}\sin^2 x + C}$

$u = \sin x$

$du = \cos x \, dx$

Ex) $\int \sin x \cos x \, dx = \int -u \, du$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= \frac{-1}{2}u^2 + C$

$= \boxed{\frac{-1}{2}\cos^2 x + C}$

same, but shifted by a constant

Remember: $\sin^2 x + \cos^2 x = 1$

$$\frac{1}{2}\sin^2 x + C = \frac{1}{2}\left(1 - \cos^2 x\right) + C = \frac{-1}{2}\cos^2 x + \frac{1}{2} + C$$

How does u-sub work w/ def. integrals?

Ex] $\int_0^1 \sin(e^x)\,e^x\,dx.$ $u=e^x$, $du=e^x\,dx$

Method ①

Remember that your bounds are in terms of x, not u.

$$\int_{x=0}^{x=1} \sin(u)\,du = -\cos(u)\Big]_{x=0}^{x=1}$$

Write antid. in terms of x before plugging in: $-\cos(e^x)\Big]_0^1 = \boxed{-\cos(e^1)+\cos(e^0)}$

Method ② Change bounds to be in terms of u, not x.

$$\int_0^1 \sin(e^x)\,e^x\,dx = \int_1^e \sin u\,du = -\cos u\Big]_1^e = \boxed{-\cos(e)+\cos(1)}$$

$u=e^x$ $du=e^x\,dx$

Bounds: $x=1 \rightarrow u=e^1=e$

$x=0 \rightarrow u=e^0=1$