Math 125D 1/12/24

Chapters 5.3 & 5.4

(and maybe some 5.5)

DFEP #1: Friday, January 12th.

Write each limit as an integral, then compute it.

(a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{i\pi}{3n}\right)\frac{\pi}{3n}$

(b) Warning: not easy! $\displaystyle\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{n^2-\frac{i^2}{4}}}$

5.3 Continued

FTC Part 1: $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

a ← constant

$$\int_a^b \sim + \int_b^c \sim = \int_a^c \sim$$

$$\int_b^c \sim = \int_a^c \sim - \int_a^b \sim$$

Ex] Find $\frac{d}{dx}\left(\int_7^x e^{(t^2)}\,dt\right) = \boxed{e^{(x^2)}}$

Ex] Find $\frac{d}{dx}\left(\int_{x^2}^{x^3} \sqrt{\sin(t)}\,dt\right)$ (If this said, find $\frac{d}{dx}\left(\int_0^x \sqrt{\sin(t)}\,dt\right)$, it would be easy: $\boxed{\sqrt{\sin x}}$)

x^2 ← not a constant!

$$\frac{d}{dx}\left(\int_{17}^{x^3} \sqrt{\sin(t)}\,dt - \int_{17}^{x^2} \sqrt{\sin(t)}\,dt\right) = \underbrace{\frac{d}{dx}\left(\int_{17}^{x^3} \sqrt{\sin t}\,dt\right) - \frac{d}{dx}\left(\int_{17}^{x^2} \sqrt{\sin t}\,dt\right)}_{\text{chain rule}}$$

$$= \boxed{\sqrt{\sin(x^3)}\,(3x^2) - \sqrt{\sin(x^2)}\,(2x)}$$

Chapter 5.4: Indefinite Integrals and Net Change

no bounds

The $\underbrace{\text{indefinite integral of } f(x) \text{ with respect to } x}$, written as $\int f(x)\,dx$, is what we'll call the general antiderivative of $f(x)$.

because it's the general antideriv.

Ex] $\int (e^x + 2x)\,dx = e^x + x^2 + C$

Why the Indefinite Integral is Bad

- **The notation is overloaded.** The Fundamental Theorem of Calculus tells us that we can compute definite integrals by using antiderivatives. But definite integrals and antiderivatives are **not** the same thing! The fact that we use the symbol $\int$ for both of them is misleading, and leads to a lot of unnecessary confusion.

- **It's weird that we call it "the" indefinite integral.** The whole point of generalized antiderivatives is that there are lots of them! Why do we use a definite article when talking about an indefinite integral?

- **It's not clear what kind of object $\int f(x)\,dx$ even is.** Is it a function? Well, not really—if it were, you could do things like plugging in specific values for x, but in most contexts it doesn't make sense to take something like $\int 3x\,dx$ and "plug in" $x = 3$. It's a *family* of functions, but this is a concept that we don't really get into until much later in math.

- **The $+C$ thing isn't even correct.** Or rather, it's fine for continuous functions, but it's technically wrong to say something like $\int \frac{1}{x}\,dx = \ln|x| + C$. You could add two different constants to the positive and negative halves of $\ln(x) + C$ and you'd still get a valid antiderivative of $\frac{1}{x}$.

We'll still use indefinite integrals and do the $+C$ business because everyone else does too, but I'll quietly wincing every time.

Reminder: Indefinite Integrals that you should know so far:

$$\int c\,dx = cx + C$$

constant

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$n \neq -1$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$$

$a > 0$ is a constant

Advice: memorize these

Net Change In 5.3, we learned that $\int_a^b f'(t)\,dt = \underbrace{f(b)-f(a)}_{\text{net change in } f \text{, from time } t=a \text{ to time } t=b}.$

So, if you integrate a rate of change of some quantity from $t=a$ to $t=b$, then you get the net change over that time period.

Ex Water is pouring into a bucket at a rate of $\frac{1}{10}\sqrt{t}$ gallons/min after t minutes.

$\underbrace{\text{In the first 30 minutes, how much water pours into the}}_{\text{what is the change in the amount of water from } t=0 \text{ to } t=30\text{?}}$ bucket?

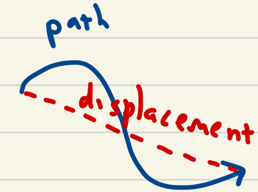
$$\int_0^{30} \frac{1}{10}\underbrace{\sqrt{t}}_{t^{1/2}}\,dt = \frac{1}{10}\left.\frac{t^{3/2}}{3/2}\right]_0^{30} = \frac{2}{30}\left(t^{3/2}\right)\Big]_0^{30} = \frac{1}{15}\left(30^{3/2}-0\right) = \frac{\cancel{30}\,2\sqrt{30}}{\cancel{15}} = \boxed{2\sqrt{30}\text{ gallons}}$$

Note: If $v(t)$ is velocity, then $\int_a^b v(t)\,dt$ represent? Displacement: change in position from $t=a$ to $t=b$

Ex] A particle moves with velocity $v(t) = t^2 - 4$ after t seconds.

Find the total distance traveled from $t=0$ to $t=5$.

2D version



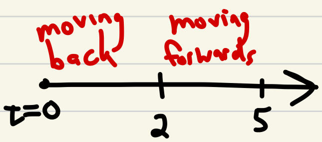
When does it turn around?

$t^2-4=0$

$t^2=4$

$t=\pm 2$

Wrong answer: $\int_0^5 (t^2-4)\,dt$. This is displacement, not total dist.



$v(t)$ is positive if $t>2$, negative if $t<2$.

Right answer: distance moving backwards + dist. moving forwards.

$$-\int_0^2 (t^2-4)\,dt + \int_2^5 (t^2-4)\,dt = -\left(\frac{t^3}{3}-4t\right)\Big]_0^2 + \left(\frac{t^3}{3}-4t\right)\Big]_2^5 = -\left(\left(\frac{8}{3}-8\right)-0\right) + \left(\left(\frac{125}{3}-20\right)-\left(\frac{8}{3}-8\right)\right)$$

$$= 16-\frac{16}{3}+\frac{125}{3}-20 = -4+\frac{109}{3}$$

$$= \boxed{\frac{97}{3}}$$

$$\int_0^5 \left|t^2-4\right|\,dt$$