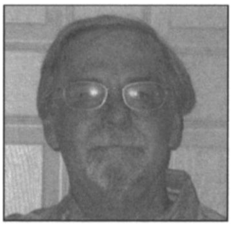
*Dinner Tables and Concentric Circles: A Harmony of Mathematics, Music, and Physics**

Jack Douthett and Richard J. Krantz



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About 20 years ago, John Clough, who at that time headed the music theory program at SUNY Buffalo, posed two related circle problems. These problems are now known as the *Dinner Table Problem* and the *Concentric Circles Problem* [**6**].

The Dinner Table Problem. Suppose one wishes to seat m men and n women around a circular dinner table. How should they be placed so that they are distributed "as evenly as possible" around the table?

The Concentric Circles Problem. Place m white points evenly around one circle, and n black points similarly around another circle of the same size. Superimpose the circles so that no two points coincide. How can the resulting distribution of points be represented?

By now you may have several questions: What is meant by "as evenly as possible"? What do these problems have in common? Why would a musician be interested in

*This paper is dedicated to the memory of John Clough (1928–2003). Without his seminal works in music theory and his patient encouragement of others, this work and much of the work referenced herein would never have been started, much less completed. The field of mathematical music theory owes a great debt to him, and the authors are privileged to have known and worked with him.

such seemingly non-musical problems? And in view of the subtitle of this article, you may also be wondering how all this relates to physics. The answers to these questions come with a theory—now known as the theory of maximally even sets—that began in 1991 with an investigation of the structure of musical scales [**5**].

Dinner tables

First, we address the meaning of "as evenly as possible." One way to define this is that a distribution is "as even as possible" if the average distance between pairs of women is as large as possible. Figure 1 shows all relevant configurations (up to rotation and reflection) for 3 women (black points) and 4 men (white points) around the unit circle. If the idea is to distribute the women as evenly as possible around the table, then the distribution in Figure 1(a) is clearly the worst possible choice, since the women are all bunched together. The average distance between women (the average of the lengths of the dotted edges) in this *minimally even distribution* is 1.10. Figures 1(b), 1(c), and 1(d) show a sequence of configurations that gradually distributes the women more and more "evenly" around the table. Since the maximum average distance between pairs of women is 1.69, Figure 1(d) is optimal.

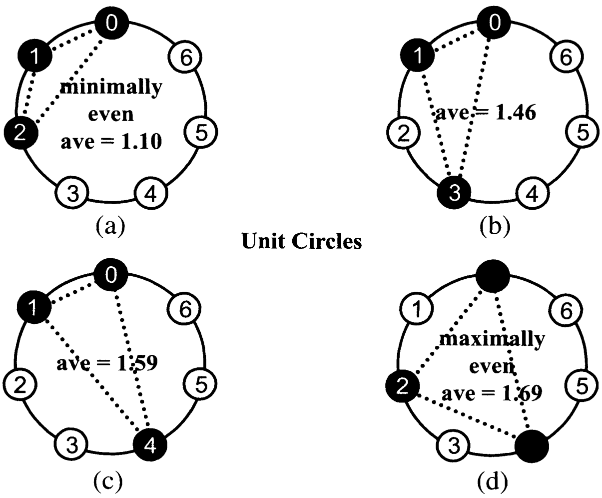


Figure 1. All relevant configurations of three women and four men.

This definition of "as evenly as possible" suggests a second possibility, that the average distance between pairs of men is maximum. Still another possible definition is that the average pair-wise distance between members of the opposite sex is minimum. But how do these definitions of "as evenly as possible" relate?

In a recent article that generalizes this problem, Douthett and Krantz [**6**] considered an s-cycle of m white and n black sites labeled consecutively with the integers $0, 1, 2, \ldots, s-1$. All paths are directed in the direction of the increasing numbers. For distinct sites x and y, the length of a path from x to y, denoted $\mathrm{dist}_s(x, y)$, is the smallest positive integer congruent to $y - x$ modulo s.

Next let J be a function, called an *interaction*, that assigns a weight to $\mathrm{dist}_s(x, y)$. If the white set $W = \{w_0, w_1, \ldots, w_{m-1}\}$ is the set of white sites, we call any path whose initial and terminal sites are in W a *white path*. The *weight of white set* W is the sum of the weights of the white path lengths,

$$\Phi_W(J, s) = \sum_{k=1}^{m-1} \sum_{j=0}^{m-1} J\big(\text{dist}_s(w_j, w_{j+k})\big) \tag{1}$$

where the subscript $j + k$ is reduced modulo m. Now assume J is a strictly convex interaction on $[1, s - 1]$. Douthett and Krantz [**6**] have shown that when the weights of all the white sets with cardinality m on s sites are compared, the sets with minimum weight are equivalent under rotation (if the interaction is convex but not *strictly* convex, there may be white sets outside this class that also have minimum weight). It is these sets that Douthett and Krantz [**6**] call *maximally even sets*. It was also shown that the complement of a maximally even set is also maximally even and that for such a configuration, the sum of the weights of the lengths of the paths whose initial and terminal sites differ in color is maximum. Since all three conditions are satisfied simultaneously, these configurations are called *maximally even configurations*.

It is not difficult to show that an optimal dinner table configuration is a special case of Douthett and Krantz's [**6**] maximally even configurations. The distance between persons seated at chairs labeled x and y at a table with s chairs is

$$\text{chord}(x, y) = 2 \sin\left(\frac{\pi \,\text{dist}_s(x, y)}{s}\right). \tag{2}$$

However, the chord connecting x and y is associated with two paths, the one from x to y and the one from y to x. So, (1) sums each chord length twice. We adjust for this by defining the *chord length interaction* as half of (2):

$$J\big(\text{dist}_s(x, y)\big) = \frac{1}{2}\text{chord}(x, y) = \sin\left(\frac{\pi \,\text{dist}_s(x, y)}{s}\right). \tag{3}$$

From (1) and (3), we get the average distance between men by summing the chord lengths and dividing by the number of chords:

$$\overline{\Phi}_W(J, s) = \frac{2}{m(m-1)}\Phi_W(J, s) = \frac{2}{m(m-1)} \sum_{k=1}^{m-1} \sum_{j=0}^{m-1} \sin\left(\frac{\pi \,\text{dist}_s(w_j, w_{j+k})}{s}\right).$$

Note that the chord length interaction in (3) is a strictly concave interaction on the interval $[1, s - 1]$. It follows that the average distance between pairs of men and between pairs of women is greatest precisely when the configuration is maximally even. Moreover, the average distance between pairs of opposite sexes is least in a maximally even configuration. This can be seen in Table 1, where the averages for all the configurations in Figure 1 are given.

Douthett and Krantz [**6**] also found a convenient way to calculate maximally even sets. Let s, m, r, and k be integers such that $1 \le m \le s$, $0 \le r \le s - 1$, and $0 \le k \le$

Table 1. Average chord lengths of the configurations in Figure 1.

Figure	1(a)	1(b)	1(c)	1(d)
Average between women	1.10	1.46	1.59	1.69
Average between men	1.28	1.46	1.52	1.57
Average between opposite sexes	1.64	1.46	1.40	1.34

$m-1$. The *J-function* with these parameters is defined as,

$$J^r_{s,m}(k) = \left\lfloor \frac{ks+r}{m} \right\rfloor$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Then the set

$$J^r_{s,m} = \left\{J^r_{s,m}(0), J^r_{s,m}(1), J^r_{s,m}(2), \ldots, J^r_{s,m}(m-1)\right\}$$

is a maximally even set of cardinality m on s sites. The symbol $J^r_{s,m}$ is called the *J-representation* of the set. Conversely, it was shown that every maximally even set has a J-representation [**6**]. The superscript r is called the *rotation index*, since it determines the rotation of the set (this index will be particularly useful when we discuss musical scales).

In Figure 1(d), $s = 7$, $m = 4$, and $r = 6$. It follows that the set of labels for chairs occupied by men is $W = J^6_{7,4} = \{1, 3, 5, 6\}$. Since the complement of a maximally even set is also maximally even, the set of labels of the chairs occupied by women also has a J-representation; that is, there exists an r, $0 \le r \le 6$, such that the set of labels for the chairs occupied by women in Figure 1(d) is $J^r_{7,3}$. That necessary value is $r = 0$. It follows that $B = J^0_{7,3} = \{0, 2, 4\}$.

Intuitively, one might expect that if s is divisible by m, then in a maximally even configuration the men are spaced evenly around the table. This is in fact the case, and it is easy to show this with the J-representations. For simplicity, let $j = s/m$, and assume that the rotation index is $r = 0$. Then

$$J^0_{s,m} = \left\{0, j, 2j, \ldots, (m-1)j\right\}.$$

It follows that the men are evenly spaced (every jth chair) around the table.

Concentric circles

For the Concentric Circles Problem, there is no interaction. So, how does this problem relate to the Dinner Table Problem? Suppose that the white and black points around the superimposed circles are chairs occupied by the two sexes around a circular table. If the chairs are adjusted so they are evenly spaced around the table, then the seating configuration is maximally even. Conversely, the evenly spaced chairs in a maximally even seating configuration can be adjusted so that same sexes are evenly spaced around the dinner table. This is illustrated in Figure 2 for three women and four men, where

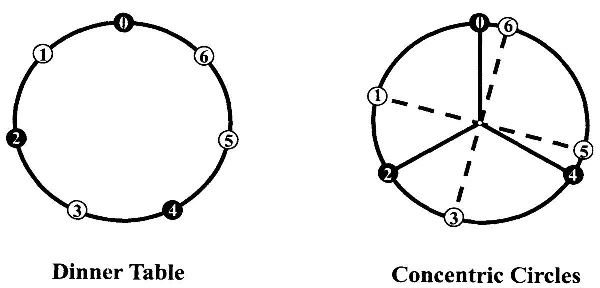


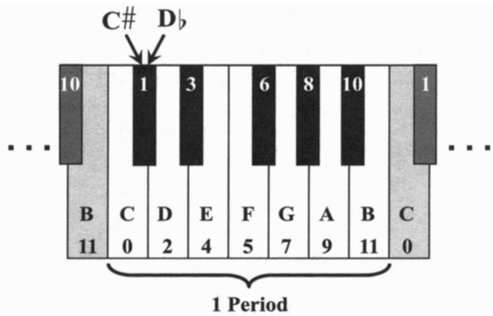
Figure 2. Dinner table and the concentric circles problems.

the dinner table is on the left the concentric circles on the right. Note that in both, the sets of labels are $W = J^6_{7,4} = \{1, 3, 5, 6\}$ and $B = J^0_{7,3} = \{0, 2, 4\}$.

As with a maximally even configuration, the white and black sets around the superimposed circles have J-representations. This was first proved by Clough and Douthett [**5**]; simpler proofs have since been found [**6**], [**10**].

The piano keyboard and diatonic scales

As we mentioned earlier, the initial work on maximally even sets was done to describe musical scale structure [**5**]. The most obvious connection between maximally even configurations and music can be seen on the piano keyboard. Figure 3 shows one period (an octave) of the pattern of white and black keys. This pattern of 12 keys repeats every octave. The white keys are given the letter names A through G, and the black keys take on the name of the left adjacent white key sharped (♯) or the right adjacent white key flatted (♭). For example, the black key between the white keys C and D can be called either C♯ (C sharp) or D♭ (D flat). In music theory, integers, called *pitch-classes*, are often assigned to the notes: 0 to the note C, 1 to C♯/D♭, 2 to D, and so forth. If the keyboard is viewed in terms of pitch-classes, then the section of the keyboard in Figure 3 is partitioned into complementary maximally even sets; the white-key set is $J^5_{12,7}$ and the black-key set $J^6_{12,5}$. These sets define musical scales known as *diatonic* and *pentatonic scales*. The union of these two scales is called the *chromatic scale*. At first glance, it may appear to some that the connection between the keyboard and maximally even sets is a coincidence. This is, however, not the case, as illustrated by Carey and Clampitt [**3**] and by Krantz and Douthett [**9**] in work that relates acoustical properties and the construction of musical scales.



$W = \{0, 2, 4, 5, 7, 9, 11\} = J^5_{12,7}$ and $B = \{1, 3, 6, 8, 10\} = J^6_{12,5}$

One octave on the piano keyboard

Figure 3. The piano keyboard as a maximally even configuration.

In addition to the white key diatonic scale, every rotation of this scale is also a diatonic scale. The class of diatonic scales can be numerically represented as a set of J-representations:

$$\{J^0_{12,7}, J^1_{12,7}, J^2_{12,7}, \ldots, J^{11}_{12,7}\}.$$

An important cycle of diatonic scales in the development of Western music is known as the *circle of fifths*. This is shown in Figure 4 where the musical names of the scales are outside the circle and their J-representations inside.

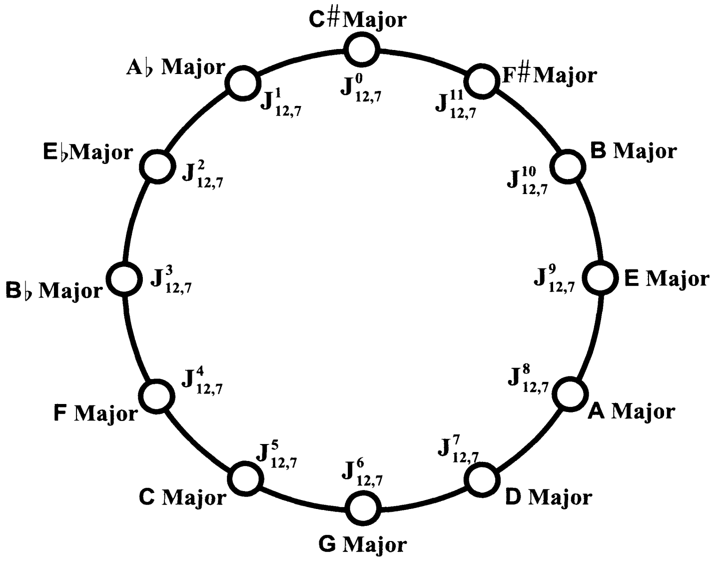


Figure 4. Circle of fifths, diatonic scales, and their J-representations.

Adjacent scales are called *closely related keys*. This is an important cycle in music because it is possible to *modulate* (move) from one scale to a closely related scale by changing a single note by a *half-step* (the smallest musical interval). With respect to J-representations, it is possible to move from one set to an adjacent set by changing one number by 1. For example, $J^5_{12,7}$ and $J^6_{12,7}$ are adjacent sets, and it is possible to move from $J^5_{12,7} = \{0, 2, 4, 5, 7, 9, 11\}$ to $J^6_{12,7} = \{0, 2, 4, 6, 7, 9, 11\}$ by changing the number 5 (in $J^5_{12,7}$) to 6 (in $J^6_{12,7}$). This relationship is true of all adjacent sets in this cycle. Moreover, two scales are closely related if, and only if, the rotation indices of their J-representations differ by 1 modulo 12. Clampitt [**4**] has shown that in general

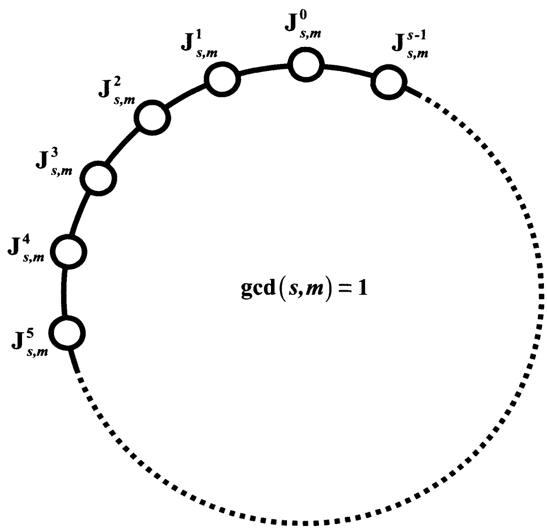


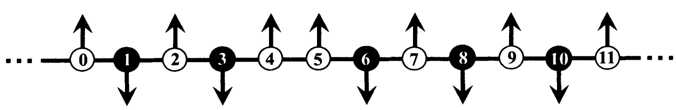
Figure 5. Generalized circle of fifths in terms of J-representations.

the sets in a class of sets equivalent under rotation are capable of forming such a cycle precisely when the sets are maximally even and s and m are coprime. Figure 5 shows this generalized circle of fifths in terms of J-representations.

Diatonic and pentatonic scales are not the only musical objects that are maximally even. For those familiar with musical scales and chords, the augmented triads ($m = 3$), fully diminished seventh chords ($m = 4$), whole-tone scales ($m = 6$), and octatonic scales ($m = 8$) are also maximally even sets (jazz musicians often refer to the octatonic scales as diminished scales). In addition, many microtonal theorists (theorists who study musical systems with other than twelve divisions to the octave) have scale constructions that are maximally even [**1**], [**2**]. These scale structures are important to many microtonal theorists precisely because of the generalized circle of fifths (see Figure 5). Those interested in a pedagogical approach to music theory based on maximally even sets should refer to Johnson's music theory text [**8**].

The 1-dimensional antiferromagnetic Ising model

The connection between maximally even configurations and physics was first observed by Douthett and Krantz [**7**]. The link appears in a physics problem known as the *1-dimensional antiferromagnetic Ising model*. This model is a 1-dimensional lattice of "particles," each having either an up-spin or a down-spin orientation. Figure 6 shows a section of such a lattice with up-arrows and down-arrows indicating the spin orientations. In the antiferromagnetic model, the up- and down-spins behave like magnets, where parallel side-by-side orientations (like spins) repel and antiparallel side-by-side orientations (opposite spins) attract. Summing the pair-wise interaction energy between spins yields the configurational energy of the system. The question is, for a given up-spin density (the ratio of up-spins to the total number of spins), what configurations yields the minimum average configurational energy? Given the subject of this article, you may have already guessed; the configurations are maximally even. Although the details are a bit more complex, this problem is similar to the Dinner Table Problem, except that now the interaction is strictly convex (see [**6**] for details). Thus, maximally even configurations minimize the average configurational energy.



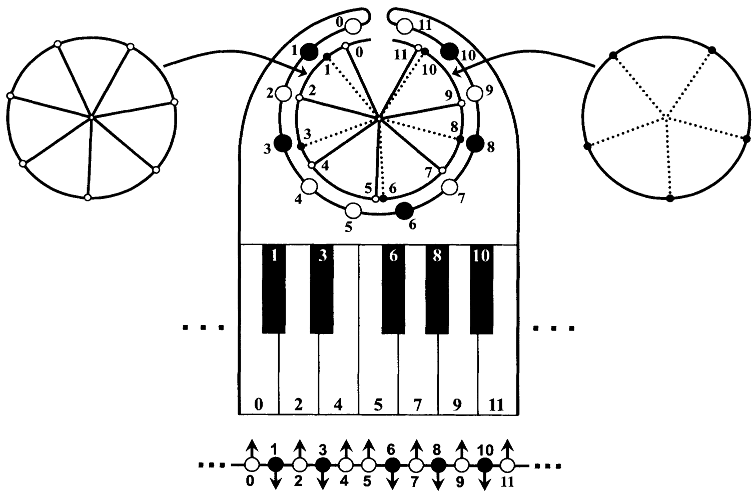
$W = \{0, 2, 4, 5, 7, 9, 11\} = J^{5}_{12,7}$ **and** $B = \{1, 3, 6, 8, 10\} = J^{6}_{12,5}$

Figure 6. Ising lattice as a maximally even configuration.

The lattice section in Figure 6 represents one period of such a configuration where the up-spin density is $7/12$. Note that the set of labels assigned to the up-spins is $J^{5}_{12,7}$, and to the down-spins is $J^{6}_{12,5}$. These are the same sets that represent the white and black keys on the piano keyboard shown in Figure 3. This means that for an up-spin density of $7/12$ and minimum configurational energy, the configuration of spins in the Ising lattice is the same as the configuration of white and black keys on the piano keyboard.

Coda

We conclude with Figure 7, which summarizes the content of this paper. The circle on the left with seven white points and the circle on the right with five black points are superimposed in the inner circle of Figure 7, which illustrates the Concentric Circles Problem. Shifting the white and black points so they are evenly spaced gives us the dinner table with seven men (white points) and five women (black points) distributed as evenly as possible (outside circle). The dinner table is then unwound, illustrating the connection between a maximally even seating configuration of seven men and five women and the white and black keys on the piano keyboard. Below the keyboard is the Ising lattice with an up-spin density of 7/12 and minimum average configurational energy that parallels the configuration of white and black keys. In each case, the configuration is maximally even, and the complementary maximally even sets partitioning the configurations are $J^5_{12,7}$ and $J^6_{12,5}$.



$$W = J^5_{12,7} = \{0, 2, 4, 5, 7, 9, 11\} \text{ and } B = J^6_{12,5} = \{1, 3, 6, 8, 10\}$$

Figure 7. Concentric circles, dinner table, piano keyboard, and the Ising lattice configurations.

It is intriguing to watch a theory born in music develop as this one has. Although it is common to apply tools developed in mathematics to physics and music, it is most unusual to borrow tools developed in music theory to explain phenomena in mathematics and physics. The theory of maximally even sets does just this. The fact that this theory can be applied to such a variety of situations suggests that it might be applied elsewhere. But where?

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One-Upmanship in Creating Designer Decimals

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In the last several years, two Classroom Capsules (34:1 (2003) 58–62 and 35:2 (2004) 125–26), an article (37:5 (2006) 355–363), and a page-filler (38:1 (2007) 46) have been published in this journal on the topic of *designer decimals*. These are fractions whose decimal expansions begin with a well-known sequence of positive integers. For example, when truncated after 194 decimal places,

10,000/9801 is 1.01 02 03 . . . 96 97.

This is a special case of the fraction $10^{2n}/(10^{2n} - 2 \cdot 10^n - 1)$.

We recently discovered that the simple operation of adding 1 repeatedly to the numerator generates nested quotients in arithmetic progressions. If we continue with our example with $n = 2$, but successively add $m = 1, 2$, and 3 to the numerator, we have the following truncated decimals:

10,001/9801 begins 1.02 04 06 . . . 94 96, and is then 99 followed by 01 03 05 . . . 93 95.

10,002/9801 begins 1.02 05 08 . . . 92 95. [After 99, it repeats. Why?]

10,003/9801 begins 1.02 06 10 . . . 90 94, then is 99 followed by 03 07 11 . . . 87 91, then 96 followed by 00 04 08 . . . 88 92, then 97 followed by 01 05 09 . . . 89 93.

Note that the differences in each arithmetic progression for each decimal representation is just $m + 1$.

We get similar results from adding 1 to the numerator for the Fibonacci family of designer fractions, $10^{2n}/(10^{2n} - 10^n - 1)$:

10,000/9899 = 1.01 02 03 05 08 13 21 34 . . .
10,001/9899 = 1.01 03 04 07 11 18 29 47 . . .
10,002/9899 = 1.01 04 05 09 14 23 37 60 . . .
10,003/9899 = 1.01 05 06 11 17 29 46 75 . . .